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Freud's Identity of Differential Geometry, the Einstein-Hilbert Equations and the Vexatious Problem of the Energy-Momentum Conservation in GR

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Abstract. We reveal in a rigorous mathematical way using the theory of differential forms, here viewed as sections of a Clifford bundle over a Lorentzian manifold, the true meaning of Freud's identity of differential geometry discovered in 1939 (as a generalization of results already obtained by Einstein in 1916) and rediscovered in disguised forms by several people. We show moreover that contrary to some claims in the literature there is not a single (mathematical) inconsistency between Freud's identity (which is a decomposition of the Einstein *indexed* 3-forms $\star\mathcal{G}^{\mathbf{a}}$ in two *gauge dependent* objects) and the field equations of General Relativity. However, as we show there is an obvious inconsistency in the way that Freud's identity is usually applied in the formulation of energy-momentum "conservation laws" in GR. In order for this paper to be useful for a large class of readers (even those ones making a first contact with the theory of differential forms) all calculations are done with all details (disclosing some of the "tricks of the trade" of the subject).

Mathematics Subject Classification (2000). Primary 83C40; Secondary 15A66.

Keywords. Freud identity, general relativity, energy-moementum conservation problem.

Contents

1. Introduction

In [24, 25, 26, 29, 30] (and in references therein) several criticisms to General Relativity (GR), continuation of the ones starting in [23], are made. It is argued there that GR is full of inconsistencies, which moreover are claimed to be solved by "isogravitation theory" [27, 29, 28]. It is not our intention here to make a detailed review of the main ideas appearing in the papers just quoted. One of our purposes here is to *prove* that a strong claim containing there, namely that the classical Freud's identity [8] of differential geometry is incompatible with the vacuum Einstein-Hilbert field equations of GR is *wrong*. We take the opportunity to recall that Freud's identity is directly related with proposals for the formulation of an energy-momentum "conservation law"[1] in GR [36]. This issue is indeed a serious and vexatious problem [22] since unfortunately, the proposals appearing in the literature are full of misconceptions. Some of them we briefly discuss below.[2]

A sample on the kind of the misconceptions associated to the interpretation of Freud's identity (and which served as inspiration for preparing the present paper) show up when we read[3], e.g., in [29] :

"A few historical comments regarding the Freud identity are in order. It has been popularly believed throughout the 20-th century that the Riemannian geometry possesses

[1] The reason for the " " will become clear soon.
[2] For more details see [21].
[3] Please, consult [29] for knowledge of the references mentioned in the quotation below.

only four identities (see, e.g., Ref. [2h]). In reality, Freud [11b][4] identified in 1939 a fourth identity that, unfortunately, was not aligned with Einstein's doctrines and, as such, the identity was ignored in virtually the entire literature on gravitation of the 20-th century.

However, as repeatedly illustrated by scientific history, structural problems simply do not disappear with their suppression, and actually grow in time. In fact, the Freud identity did not escape Pauli who quoted it in a footnote of his celebrated book of 1958 [2g][5]. Santilli became aware of the Freud identity via an accurate reading of Pauli's book (including its important footnotes) and assumed the Freud identity as the geometric foundation of the gravitational studies presented in Ref. [7d]. Subsequently, in his capacity as Editor in Chief of Algebras, Groups and Geometries, Santilli requested the mathematician Hanno Rund, a known authority in Riemannian geometry [2i], to inspect the Freud identity for the scope of ascertaining whether the said identity was indeed a new identity. Rund kindly accepted Santilli's invitation and released paper [11c] of 1991 (the last paper prior to his departure) in which Rund confirmed indeed the character of Eqs. (3.10) as a genuine, independent, fourth identity of the Riemannian geometry.

The Freud identity was also rediscovered by Yilmaz (see Ref. [11d][6] and papers quoted therein) who used the identity for his own broadening of Einstein's gravitation via an external stress-energy tensor that is essentially equivalent to the source tensor with non-null trace of Ref. [11a], Eqs.3.6. Despite these efforts, the presentation of the Freud identity to various meetings and several personal mailings to colleagues in gravitation, the Freud identity continues to be main vastly ignored to this day, with very rare exceptions (the indication by colleagues of additional studies on the Freud identify not quoted herein would be gratefully appreciated)."

The paper is organized as follows. In Section 2 we present some preliminaries which fix our notations and serve the purpose to present the Einstein-Hilbert equations of GR within the theory of differential forms, something that makes transparent the nature of all the objects involved. In Section 3 we recall the Einstein-Hilbert Lagrangian density $\mathcal{L}_{EH}$ and the first order gravitational Lagrangian $\mathcal{L}_g$ and the resulting field equations. In Section 4 we recall that the components of the "2-forms" $\star\mathcal{S}_\mu$ (Eq.(34)) differs by[7] $\sqrt{-\mathbf{g}}$ from the components of the objects $\mathfrak{U}_\mu^{\rho\sigma}$ (Eq.(113)) defined by Freud. We then explicitly show that there is no incompatibility between Einstein equations and Freud's identity which is seen as a gauge dependent decomposition of the Einstein 3-forms $\star\mathcal{G}_\mu$. In Section 5 we recall a real tragic problem, namely that there are *no* genuine conservation laws of energy-momentum (and of course angular momentum) in GR. Now, the details of the proofs in Section 4 are presented in details in the Appendix C, and as the reader will see, is an arduous exercise on the algebra and calculus of the theory of differential forms, mathematical objects which in this paper are supposed to be sections of the Clifford bundle of differential forms over a Lorentzian manifold. A

[4] Reference [8] in the present paper.
[5] Reference [17] in the present paper.
[6] Reference [42] in the present paper.
[7] See Eq.(70) for the definition of $\sqrt{-\mathbf{g}}$.

summary of the main results of the Clifford bundle formalism, containing the main identities necessary for the purposes of the present paper is given in Appendix A.[8]

2. Preliminaries

A Lorentzian manifold structure is a triple $\mathbf{L} = (M, \mathbf{g}, \tau_g)$ where M is a *real* 4-dimensional manifold (which is Hausdorff, paracompact, connected and non-compact), equipped with a Lorentzian metric $\mathbf{g} \in \sec T_0^2 M$ and oriented by $\tau_g \in \sec \bigwedge^4 T^* M$.

A spacetime structure is a pentuple $\mathfrak{M} = (M, \mathbf{g}, D, \tau_g, \uparrow)$ where $(M, \mathbf{g}, \tau_g)$ is a Lorentzian manifold, D is the Levi-Civita connection of $\mathbf{g}$ and $\uparrow$ is an equivalence in $\mathbf{L}$ defining time orientation.[9]

It is well known that in Einstein's General Relativity Theory (GRT) each gravitational field generated by an energy-momentum density $\mathbf{T} \in \sec T_0^2 M$ is modelled by an appropriate $\mathfrak{M}$ [21, 22] .

Once $\mathbf{T} \in \sec T_0^2 M$ is given the field $\mathbf{g}$ is determined through Einstein equation[10],

$$\mathbf{G} = \mathbf{Ric} - \frac{1}{2}\mathbf{g}R = -\mathbf{T} = \mathbf{T}, \tag{1}$$

where $\mathbf{Ric} \in \sec T_0^2 M$ is the *Ricci* tensor, R is the *curvature scalar* and $\mathbf{G} \in \sec T_0^2 M$ is the *Einstein* tensor.

Let (φ, U) be a chart for $U \subset M$ with coordinates $\{x^\mu\}$. A coordinate basis for TU is $\{\partial_\mu = \frac{\partial}{\partial x^\mu}\}$ and its *dual basis* (i.e., a basis for T^*U) is $\{\boldsymbol{\gamma}^\mu = dx^\mu\}$. We introduce also an orthonormal basis $\{\mathbf{e_a}\}$ for TU and corresponding dual basis $\{\boldsymbol{\theta}^\mathbf{a}\}$ for T^*U.

We have

$$\begin{gathered}\mathbf{e_a} = h_\mathbf{a}^\mu \partial_\mu, \quad \boldsymbol{\theta}^\mathbf{a} = h_\mu^\mathbf{a} dx^\mu, \\ h_\mu^\mathbf{a} h_\mathbf{b}^\mu = \delta_\mathbf{b}^\mathbf{a}, \quad h_\nu^\mathbf{a} h_\mathbf{a}^\mu = \delta_\nu^\mu.\end{gathered} \tag{2}$$

The metric field is expressed in those bases as,

$$\begin{gathered}\mathbf{g} = g_{\mu\nu} \boldsymbol{\gamma}^\mu \otimes \boldsymbol{\gamma}^\nu, \\ \mathbf{g} = \eta_\mathbf{ab} \boldsymbol{\theta}^\mathbf{a} \otimes \boldsymbol{\theta}^\mathbf{b},\end{gathered} \tag{3}$$

where the matrix with entries $\eta_\mathbf{ab}$ is $\mathrm{diag}(1, -1, -1, -1)$.

Next we introduce a metric $\mathtt{g} \in \sec T_2^0 M$ on the cotangent bundle by:

$$\begin{gathered}\mathtt{g} = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^v}, \\ \mathtt{g} = \eta^\mathbf{ab} \mathbf{e_a} \otimes \mathbf{e_b},\end{gathered} \tag{4}$$

where the matrix with entries $\eta^\mathbf{ab}$ is $\mathrm{diag}(1, -1, -1, -1)$ and $g^{\mu\nu} g_{\nu\lambda} = \delta_\lambda^\mu$.

[8] A detailed presentation of the subject may be found in [21].
[9] Details may be found, e.g., in [21, 22].
[10] We use natural units.

We introduce also the *reciprocal basis* of $\{\partial_\mu = \frac{\partial}{\partial x^\mu}\}$ and of $\{\mathbf{e_a}\}$ as being respectively the basis $\{\partial^\mu\}$ and $\{\mathbf{e}^a\}$ for TU such that

$$\mathbf{g}(\partial_\mu, \partial^\nu) = \delta_\mu^\nu,\ \mathbf{g}(\mathbf{e_a}, \mathbf{e^b}) = \delta_\mathbf{a}^\mathbf{b}, \tag{5}$$

and the *reciprocal basis* of $\{\boldsymbol{\gamma}^\mu = dx^\mu\}$ and $\{\boldsymbol{\theta}^\mathbf{a}\}$ as being respectively the basis $\{\boldsymbol{\gamma}_\mu\}$ and $\{\boldsymbol{\theta}_\mathbf{a}\}$ for T^*U such that

$$\mathtt{g}(\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}^\nu) = \delta_\mu^\nu,\ \mathtt{g}(\boldsymbol{\theta}_\mathbf{a}, \boldsymbol{\theta}^\mathbf{b}) = \delta_\mathbf{a}^\mathbf{b}. \tag{6}$$

We now observe that $\mathbf{Ric}, \mathbf{g}$ and $\mathbf{T}$ (and of course, also $\mathbf{T}$) can be considered 1-form valued 1-form fields, i.e., we can write

$$\begin{aligned}
\mathbf{Ric} &= \mathcal{R}_\mu \otimes \boldsymbol{\gamma}^\mu = \mathcal{R}^\mu \otimes \boldsymbol{\gamma}_\mu = \mathcal{R}_\mathbf{a} \otimes \boldsymbol{\theta}^\mathbf{a} = \mathcal{R}^\mathbf{a} \otimes \boldsymbol{\theta}_\mathbf{a},\\
\mathbf{g} &= \boldsymbol{\gamma}_\mu \otimes \boldsymbol{\gamma}^\mu = \boldsymbol{\gamma}^\mu \otimes \boldsymbol{\gamma}_\mu = \boldsymbol{\theta}_\mathbf{a} \otimes \boldsymbol{\theta}^\mathbf{a} = \boldsymbol{\theta}^\mathbf{a} \otimes \boldsymbol{\theta}_\mathbf{a},\\
\mathbf{T} &= \mathbf{T}_\mu \otimes \boldsymbol{\gamma}^\mu = \mathbf{T}^\mu \otimes \boldsymbol{\gamma}_\mu = \mathbf{T}_\mathbf{a} \otimes \boldsymbol{\theta}^\mathbf{a} = \mathbf{T}^\mathbf{a} \otimes \boldsymbol{\theta}_\mathbf{a},
\end{aligned} \tag{7}$$

where the $\mathcal{R}_\mu = R_{\mu\nu}\boldsymbol{\gamma}^\nu \in \sec \bigwedge^1 T^*M$ (or the $\mathcal{R}^\mu \in \sec \bigwedge^1 T^*M$ or the $\mathcal{R}_\mathbf{a} \in \sec \bigwedge^1 T^*M$ or the $\mathcal{R}^\mathbf{a} \in \sec \bigwedge^1 T^*M$) are called the Ricci 1-form fields and the $\mathbf{T}_\mu = \mathbf{T}_{\mu\nu}\boldsymbol{\gamma}^\nu \in \sec \bigwedge^1 T^*M$ (or the $\mathbf{T}^\mu \in \sec \bigwedge^1 T^*M$ or the $\mathbf{T}_\mathbf{a} \in \sec \bigwedge^1 T^*M$ or the $\mathbf{T}^\mathbf{a} \in \sec \bigwedge^1 T^*M$) are called the (negative) energy-momentum 1-form fields.[11]

We also introduce the Einstein 1-form fields $(\mathcal{G}_\mu, \mathcal{G}^\mu, \mathcal{G}_\mathbf{a}, \mathcal{G}^\mathbf{a})$ by writing the Einstein tensor as

$$\mathbf{G} = \mathcal{G}_\mu \otimes \boldsymbol{\gamma}^\mu = \mathcal{G}^\mu \otimes \boldsymbol{\gamma}_\mu = \mathcal{G}_\mathbf{a} \otimes \boldsymbol{\theta}^\mathbf{a} = \mathcal{G}^\mathbf{a} \otimes \boldsymbol{\theta}_\mathbf{a}, \tag{8}$$

where, e.g.,

$$\mathcal{G}_\mu = G_{\mu\nu}\boldsymbol{\gamma}^\nu,\ \mathcal{G}_\mathbf{a} = G_\mathbf{ab}\boldsymbol{\theta}^\mathbf{b},\ \text{etc.}\ldots \tag{9}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{10}$$

We now write Einstein equation (Eq.(1)) as a set of equations for the Einstein 1-form fields, i.e.

$$\mathcal{G}_\mu = \mathbf{T}_\mu \text{ or } \mathcal{G}_\mathbf{a} = \mathbf{T}_\mathbf{a}. \tag{11}$$

We denoted by $\star$ the Hodge dual operator and write the dual of Eq.(11) as

$$\star\mathcal{G}_\mu = \star\mathbf{T}_\mu \text{ or } \star\mathcal{G}_\mathbf{a} = \star\mathbf{T}_\mathbf{a}. \tag{12}$$

[11] Keep in mind that $\mathbf{T} = -T_\nu^\mu\ \partial_\mu \otimes \boldsymbol{\gamma}^\nu$ and $\mathbf{T} = T_\nu^\mu \partial_\mu \otimes \boldsymbol{\gamma}^\nu$.

3. Gravitational Lagragian Densities

As it is well known the Einstein-Hilbert Lagrangian density is

$$\mathfrak{L}_{EH} = \frac{1}{2} \star R = \frac{1}{2} R\tau_{\mathbf{g}}. \tag{13}$$

We can easily verify that $\mathfrak{L}_{EH}$ can be written, e.g., as:

$$\mathfrak{L}_{EH} = \frac{1}{2}\mathcal{R}_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \tag{14}$$

where

$$\mathcal{R}^{\mathbf{c}}_{\mathbf{d}} = d\omega^{\mathbf{c}}_{\mathbf{d}} + \omega^{\mathbf{c}}_{\mathbf{k}} \wedge \omega^{\mathbf{k}}_{\mathbf{d}} \in \sec \bigwedge^2 T^*M \tag{15}$$

are the *curvature* 2-form fields with each one of the $\mathcal{R}^{\mathbf{c}}_{\mathbf{d}}$ being given by

$$\mathcal{R}^{\mathbf{c}}_{\mathbf{d}} = \frac{1}{2} R_{\mathbf{d}\ \mathbf{kl}}^{\ \mathbf{c}} \boldsymbol{\theta}^{\mathbf{k}} \wedge \boldsymbol{\theta}^{\mathbf{l}}, \tag{16}$$

where $R_{\mathbf{d}\ \mathbf{kl}}^{\ \mathbf{c}}$ are the components of the Riemann tensor in the orthogonal basis and where the $\omega^{\mathbf{c}}_{\mathbf{d}} := \omega^{\mathbf{c}}_{\mathbf{ad}}\boldsymbol{\theta}^{\mathbf{a}}$ are the the connection 1-forms in the gauge defined by the orthonormal bases $\{\mathbf{e_a}\}$ and $\{\boldsymbol{\theta}^{\mathbf{a}}\}$, i.e., $D_{\mathbf{e_a}}\boldsymbol{\theta}^{\mathbf{b}} = -\omega^{\mathbf{b}}_{\mathbf{ac}}\boldsymbol{\theta}^{\mathbf{c}}$.

We recall that this paper the components of the Ricci tensor are defined according to the following convention [3, 19]

$$\mathbf{Ric} = R_{\mathbf{d}\ \mathbf{ka}}^{\ \mathbf{a}} \boldsymbol{\theta}^{\mathbf{d}} \otimes \boldsymbol{\theta}^{\mathbf{k}}. \tag{17}$$

We recall moreover that Eq.(15) is called Cartan's second structure equation (valid for an arbitrary connection). Cartan's first structure equations reads (in an orthonormal basis) for a torsion free connection, which is the case of a Lorentzian spacetime

$$d\boldsymbol{\theta}^{\mathbf{a}} = -\omega^{\mathbf{a}}_{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{b}}. \tag{18}$$

Also, it is not well known *as it should be* that $\mathfrak{L}_{EH}$ can be written as[12]:

$$\mathfrak{L}_{EH} = \mathfrak{L}_g - d(\boldsymbol{\theta}^{\mathbf{a}} \wedge \star d\boldsymbol{\theta}^{\mathbf{a}}) \tag{19}$$

with[13]

$$\mathfrak{L}_g = -\frac{1}{2} d\boldsymbol{\theta}^{\mathbf{a}} \wedge \star d\boldsymbol{\theta}_{\mathbf{a}} + \frac{1}{2}\delta\boldsymbol{\theta}^{\mathbf{a}} \wedge \star\delta\boldsymbol{\theta}_{\mathbf{a}} + \frac{1}{4}(d\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}_{\mathbf{a}}) \wedge \star(d\boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}_{\mathbf{b}}), \tag{20}$$

where δ is the Hodge coderivative operator.

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[12] Details may be found in [21].

[13] An equivalent expression for $\mathcal{L}_g(\boldsymbol{\theta}^{\mathbf{a}}, d\boldsymbol{\theta}^{\mathbf{a}})$ is given in [37]. However the formula there does not disclose that $\mathcal{L}_g$ contains a Yangs-Mill term, a gauge fixing term and an auto interaction term (in the form of interaction of the vorticities of the fields $\boldsymbol{\theta}^{\mathbf{a}}$), something that suggests according to us, a more realistic interpretation of Einstein's gravitational theory, i.e., as a theory of physical fields in the Faraday sense living and interacting with all matter fields in Minkowski spacetime [16].

Then the total Lagragian for the gravitational plus matter field can be written as

$$\mathfrak{L} = \mathfrak{L}_g + \mathfrak{L}_m, \tag{21}$$

where due to the principle of minimal coupling $\mathfrak{L}_m$ depends on the matter fields (represented by some differential forms[14]) and the $\boldsymbol{\theta}^{\mathbf{a}}$ (due to the use of the Hodge dual in the writing of $\mathfrak{L}_m$).

The variational principle

$$\boldsymbol{\delta} \int (\mathfrak{L}_{EH} + \mathfrak{L}_m) = 0, \tag{22}$$

or

$$\boldsymbol{\delta} \int (\mathfrak{L}_g + \mathfrak{L}_m) = 0, \tag{23}$$

then must give with the usual hypothesis that the boundary terms are null the *same* equations of motion. From Eq.(22) we get supposing that $\mathfrak{L}_m$ does not depend explicitly on the $d\boldsymbol{\theta}_{\mathbf{a}}$ (principle of minimal coupling) that

$$\begin{aligned} &\int \boldsymbol{\delta}(\mathfrak{L}_{EH} + \mathfrak{L}_m) \\ &= \int (\boldsymbol{\delta}\mathfrak{L}_{EH} + \boldsymbol{\delta\theta}^{\mathbf{a}} \wedge \frac{\partial \mathfrak{L}_m}{\partial \boldsymbol{\theta}_{\mathbf{a}}}). \end{aligned} \tag{24}$$

The result of this variation is (see details in the Appendix B):

$$\begin{aligned} \int \boldsymbol{\delta}(\mathfrak{L}_{EH} + \mathfrak{L}_m) &= \int \boldsymbol{\delta\theta}^{\mathbf{a}} \wedge (\frac{\boldsymbol{\delta}\mathfrak{L}_{EH}}{\boldsymbol{\delta\theta}^{\mathbf{a}}} + \frac{\partial \mathfrak{L}_m}{\partial \boldsymbol{\theta}^{\mathbf{a}}}) \\ &= \int \boldsymbol{\delta\theta}^{\mathbf{a}} \wedge (-\star \mathcal{G}_{\mathbf{a}} + \frac{\partial \mathfrak{L}_m}{\partial \boldsymbol{\theta}^{\mathbf{a}}}) = 0. \end{aligned} \tag{25}$$

and the equations of motion are:

$$\mathcal{R}^{\mathbf{a}} - \frac{1}{2} R \boldsymbol{\theta}^{\mathbf{a}} = -\mathbf{T}^{\mathbf{a}}. \tag{26}$$

In Eq.(26) the $\mathcal{G}^{\mathbf{a}} = \left(\mathcal{R}^{\mathbf{a}} - \frac{1}{2} R \boldsymbol{\theta}^{\mathbf{a}}\right) \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$, $\mathcal{R}^{\mathbf{a}} = R^{\mathbf{a}}_{\mathbf{b}} \boldsymbol{\theta}^{\mathbf{b}} \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$, R and the $\mathbf{T}^{\mathbf{a}}$ have already been given names. We moreover have:

$$\star \mathbf{T}^{\mathbf{a}} = - \star \mathbf{T}^{\mathbf{a}} := -\frac{\partial \mathfrak{L}_m}{\partial \boldsymbol{\theta}^{\mathbf{a}}} \in \sec \bigwedge^3 T^*M, \tag{27}$$

as the definition of the energy-momentum 3-forms of the matter fields.

We now have an important result, need for one of the purposes of the present paper.

Theorem. *The* $\star\mathcal{G}^{\mathbf{a}} \in \sec \bigwedge^3 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$ *can be written:*

$$-\star \mathcal{G}^{\mathbf{a}} = d \star \mathcal{S}^{\mathbf{a}} + \star t^{\mathbf{a}}, \tag{28}$$

[14]We emphasize that the present formalism is applicable even to spinor fields, which as proved in [15, 21] can safely be represented by appropriate classes of *non homogeneous* differential forms.

with

$$\star\mathcal{S}^{\mathbf{c}} = \frac{1}{2}\omega_{\mathbf{ab}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{c}}) \in \sec \bigwedge\nolimits^{2} T^{*}M \tag{29}$$

$$\star t^{\mathbf{c}} = -\frac{1}{2}\omega_{\mathbf{ab}} \wedge [\omega_{\mathbf{d}}^{\mathbf{c}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) + \omega_{\mathbf{d}}^{\mathbf{b}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{d}} \wedge \boldsymbol{\theta}^{\mathbf{c}})] \tag{30}$$
$$\in \sec \bigwedge\nolimits^{3} T^{*}M.$$

The $\star\mathcal{S}^{\mathbf{c}} \in \sec \bigwedge^{2} T^{*}M$ are called *superpotentials* and the $\star t^{\mathbf{c}}$ are called the *gravitational energy-momentum pseudo* 3- *forms*. The reason for this name is given in Remark **1**.

Proof. To proof the theorem we compute $-2\star\mathcal{G}^{\mathbf{a}}$ as follows:

$$\begin{aligned}
-2\star\mathcal{G}^{\mathbf{d}} &= d\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) + \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&= d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] + \boldsymbol{\omega}_{\mathbf{ab}} \wedge d\star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&+ \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&= d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] - \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{a}} \wedge \star(\boldsymbol{\theta}^{\mathbf{p}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&- \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{b}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{p}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) - \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{d}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{p}}) \\
&+ \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&= d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] - \boldsymbol{\omega}_{\mathbf{ab}} \wedge [\boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{d}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}^{\mathbf{p}}) \\
&+ \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{b}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{p}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] \\
&= 2(d\star\mathcal{S}^{\mathbf{d}} + \star t^{\mathbf{d}}).
\end{aligned} \tag{31}$$

□

So, we just showed that Einstein equations can be written in the suggestive form:

$$-d\star\mathcal{S}^{\mathbf{a}} = (\star\mathbf{T}^{\mathbf{a}} + \star t^{\mathbf{a}}), \tag{32}$$

which implies the differential conservation law $d(\star\mathbf{T}^{\mathbf{a}} + \star t^{\mathbf{a}}) = 0$, to be scrutinized below. We start, with the

Remark 1. The $\star t^{\mathbf{a}}$ are not true *index* 3-forms [21], i.e., there do not exist a tensor field $\mathbf{t} \in \sec T_{1}^{3}M$ such that for $v_i \in \sec TM$, $i = 1, 2, 3$,

$$\mathbf{t}(v_1, v_2, v_3, \boldsymbol{\theta}^{\mathbf{a}}) = \star t^{\mathbf{a}}(v_1, v_2, v_3). \tag{33}$$

We can immediately understand why this is the case, if we recall the dependence of the $\star t^{\mathbf{a}}$ on the connection 1-forms and that these objects are *gauge dependent* and thus do not transform homogeneously under a change of orthonormal frame. Equivalently, the set $t_{\mathbf{cd}}$ ($t_{\mathbf{c}} = t_{\mathbf{cd}}\boldsymbol{\theta}^{\mathbf{d}}$) for $\mathbf{c}, \mathbf{d} = 0, 1, 2, 3$ are *not* the components of a tensor field. So, these components are said to define a *pseudo-tensor*.

The $\star\mathcal{S}^{\mathbf{a}}$ also are not true index forms for the same reason as the $\star t^{\mathbf{a}}$, they are gauge dependent.

Remark 2. Eq.(32) is known in recent literature of GR as Sparling equations [34] because it appears (in an equivalent form) in a preprint [32] of 1982 by that author. However, it already appeared early, e.g., in a 1978 paper by Thirring and Wallner [35].

Remark 3. We emphasize that if we had used a coordinate basis we would get analogous equations, i.e.

$$-d\star\mathcal{S}^{\rho}=(\star\mathbf{T}^{\rho}+\star t^{\rho}),\tag{34}$$

$$\begin{aligned}\star\mathcal{S}^{\rho}&=\frac{1}{2}\Gamma_{\alpha\beta}\wedge\star(\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}^{\beta}\wedge\boldsymbol{\gamma}^{\rho})\in\sec\bigwedge\nolimits^{2}T^{*}M\\ \star t^{\rho}&=-\frac{1}{2}\Gamma_{\alpha\beta}\wedge[\Gamma^{\rho}_{\sigma}\wedge\star(\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}^{\beta}\wedge\boldsymbol{\gamma}^{\sigma})+\Gamma^{\beta}_{\sigma}\wedge\star(\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}^{\rho}\wedge\boldsymbol{\gamma}^{\sigma})]\\ &\in\sec\bigwedge\nolimits^{3}T^{*}M,\end{aligned}\tag{35}$$

with the 1-form of connections given by $\boldsymbol{\Gamma}^{\rho}_{\sigma}:=\Gamma^{\rho}_{\alpha\sigma}\boldsymbol{\gamma}^{\alpha}$, $D_{\partial_\sigma}\boldsymbol{\gamma}^{\rho}=-\Gamma^{\rho}_{\alpha\sigma}\boldsymbol{\gamma}^{\alpha}$.

Note that Eq.(34), e.g., shows that each one of the 2-form fields $\star\mathcal{S}^{\mu}$ (*the superpotentials*) is only defined modulo a closed 2-form $\star N^{\mu}$, $d\star N^{\mu}=0$.

Remark 4. The use of a pseudo-tensor to express the conservation law of energy-momentum of matter plus the gravitational field appeared in a 1916 paper by Einstein [5]. His pseudo-tensor which has been originally presented in a coordinate basis is identified (using the works of [13] and [36]) in the Appendix D. We show that Einstein's superpotentials are the Freud's "2 -forms" $\star\mathbf{U}^{\lambda}$ (Eq.(127)).

Remark 5. We now turn to $\boldsymbol{\delta}\int(\mathfrak{L}_g+\mathfrak{L}_m)=0$. We immediately get

$$\int\boldsymbol{\delta\theta}^{\mathbf{a}}\wedge\left[\frac{\partial\mathcal{L}_g}{\partial\boldsymbol{\theta}_a}+d\left(\frac{\partial\mathcal{L}_g}{\partial d\boldsymbol{\theta}_{\mathbf{a}}}\right)+\frac{\partial\mathfrak{L}_m}{\partial\boldsymbol{\theta}_{\mathbf{a}}}\right].\tag{36}$$

The computation of $\frac{\partial\mathcal{L}_g}{\partial\boldsymbol{\theta}_a}$ and $d\left(\frac{\partial\mathcal{L}_g}{\partial d\boldsymbol{\theta}_{\mathbf{a}}}\right)$ is a very long one and will not be given in this paper. However, of course, we get:

$$-\star\mathcal{G}^{\mathbf{a}}=\frac{\partial\mathcal{L}_g}{\partial\boldsymbol{\theta}_a}+d\left(\frac{\partial\mathcal{L}_g}{\partial d\boldsymbol{\theta}_{\mathbf{a}}}\right)=\star\underset{g}{t}^{\mathbf{a}}+d\star\underset{g}{\mathcal{S}}^{\mathbf{a}}=-\star\mathbf{T}^{\mathbf{a}},\tag{37}$$

and a detailed calculation (see details in [21]) gives:

$$\begin{aligned}\star\underset{g}{\mathcal{S}}^{\mathbf{a}}&=\frac{\partial\mathcal{L}_g}{\partial d\boldsymbol{\theta}_{\mathbf{a}}}=\star\mathcal{S}^{\mathbf{a}},\\ \star\underset{g}{t}^{\mathbf{a}}&=\frac{\partial\mathcal{L}_g}{\partial\boldsymbol{\theta}_a}.=\star t^{\mathbf{a}}.\end{aligned}\tag{38}$$

4. Freud's Identity

To compute the components of the $\mathcal{S}_\mu = \frac{1}{2}\mathcal{S}_\mu^{\nu\rho}\boldsymbol{\gamma}_\nu \wedge \boldsymbol{\gamma}_\rho \in \sec \bigwedge^2 T^*M$ is a trick exercise on the algebra of differential forms. For that reason we give the details in the Appendix C, where using the techniques of Clifford bundle formalism we found directly that

$$\mathcal{S}_\mu^{\lambda\rho} = \frac{1}{2}\det\begin{bmatrix} \delta_\mu^\lambda & \delta_\mu^\sigma & \delta_\mu^\iota \\ g^{\lambda\kappa} & g^{\sigma\kappa} & g^{\iota\kappa} \\ \Gamma_{\kappa\iota}^\lambda & \Gamma_{\kappa\iota}^\sigma & \Gamma_{\iota\kappa}^\iota \end{bmatrix}, \tag{39}$$

which we moreover show to be equivalent to[15]

$$\mathcal{S}_\mu^{\nu\rho} = \frac{1}{2\sqrt{-\mathbf{g}}}\mathfrak{g}_{\mu\sigma}\partial_\beta(\mathfrak{g}^{\nu\beta}\mathfrak{g}^{\sigma\rho} - \mathfrak{g}^{\rho\beta}\mathfrak{g}^{\sigma\nu}), \tag{40}$$

with the definition of $\mathfrak{g}_{\mu\sigma}$ and $\mathfrak{g}^{\nu\beta}$ given in Eq.(114) and $\mathbf{g}$ in Eq.(70) (Appendix A.1.1).

From Eq.(39) we immediately see (from the last formula in Freud's paper [8]) that the object that he called $\mathfrak{U}_\mu^{\nu\rho}$ must be identified with

$$\mathfrak{U}_\mu^{\nu\rho} = \sqrt{-\mathbf{g}}\mathcal{S}_\mu^{\nu\rho}, \tag{41}$$

and the one he called $\mathfrak{U}_\mu^\nu$ (Eq.(1) of [8]) is

$$\mathfrak{U}_\mu^\nu = \frac{\partial}{\partial x^\rho}\mathfrak{U}_\mu^{\nu\rho}. \tag{42}$$

The $\mathfrak{U}_\mu^{\nu\rho}$ are the *superpotentials* appearing in Freud's classical paper and, of course,

$$\frac{\partial}{\partial x^\nu}\mathfrak{U}_\mu^\nu = 0. \tag{43}$$

With the above identifications we verified in the Appendix that the identity derived above (see Eq.(37))

$$\mathcal{G}^\iota = -t^\iota - \star^{-1}d\star\mathcal{S}^\iota \tag{44}$$

is equivalent to

$$\begin{aligned} 2\mathfrak{U}_\kappa^\iota &= \delta_\kappa^\iota\{\sqrt{-\mathbf{g}}[R + g^{\mu\nu}\left(\Gamma_{\mu\sigma}^\rho\Gamma_{\rho\nu}^\sigma - \Gamma_{\mu\nu}^\rho\Gamma_{\rho\sigma}^\sigma\right)]\} - 2\sqrt{-\mathbf{g}}R_\kappa^\iota \\ &+ \left(\Gamma_{\mu\nu}^\iota\partial_\varkappa(\sqrt{-\mathbf{g}}g^{\mu\nu}) - \Gamma_{\mu\nu}^\nu\partial_\varkappa(\sqrt{-\mathbf{g}}g^{\mu\iota})\right), \end{aligned} \tag{45}$$

which is Eq.(8) in Freud's paper (Freud's identity) [8].

In several papers and books [25, 26, 29, 30, 28] Santilli claims that Einstein's gravitation in vacuum ($G_\nu^\mu = 0$) is incompatible with the Freud identity of Riemannian geometry.

To endorse his claim, first Santilli printed a version of Freud's identity, i.e., his Eq.(3.10) in [30] (or in Eq.(1.4.10) in [29]) with a *missing term*, as we now

[15] We observe that Eq.(40) has also been found, e.g., in [35, 36].

show. Indeed, putting $\mathfrak{R}^\iota_\kappa = \sqrt{-\mathbf{g}}R^\iota_\kappa$, $\mathfrak{R} = \sqrt{-\mathbf{g}}R$ and recalling the definition of $\mathfrak{L}$ in Eq.(134), we can rewrite Eq.(45) as:

$$\begin{aligned}&\mathfrak{R}^\iota_\kappa - \frac{1}{2}\delta^\iota_\kappa\mathfrak{R} - \frac{1}{2}\delta^\iota_\kappa\mathfrak{L}\\ &= \frac{1}{2}(\Gamma^\iota_{\mu\nu}\partial_\varkappa\mathfrak{g}^{\mu\nu} - \Gamma^\nu_{\mu\nu}\partial_\varkappa\mathfrak{g}^{\mu\iota}) - \mathfrak{U}^\iota_\kappa,\end{aligned} \tag{46}$$

Now, we can easily verify the identity[16]:

$$\frac{1}{2}(\Gamma^\iota_{\mu\nu}\partial_\varkappa\mathfrak{g}^{\mu\nu} - \Gamma^\nu_{\mu\nu}\partial_\varkappa\mathfrak{g}^{\mu\iota}) = -\frac{1}{2}\frac{\partial\mathfrak{L}}{\partial(\partial_\iota g^{\mu\nu})}\partial_\varkappa g^{\mu\nu}, \tag{47}$$

which permits us to write

$$\begin{aligned}&\mathfrak{R}^\iota_\kappa - \frac{1}{2}\delta^\iota_\kappa\mathfrak{R} - \frac{1}{2}\delta^\iota_\kappa\mathfrak{L}\\ &= -\frac{1}{2}\frac{\partial\mathfrak{L}}{\partial(\partial_\iota g^{\mu\nu})}\partial_\varkappa g^{\mu\nu} - \mathfrak{U}^\iota_\kappa.\end{aligned} \tag{48}$$

This equation can also be written, (with $\mathfrak{L} = \sqrt{-\mathbf{g}}\Theta$):

$$\begin{aligned}&R^\iota_\kappa - \frac{1}{2}\delta^\iota_\kappa R - \frac{1}{2}\delta^\iota_\kappa\Theta\\ &= -\frac{1}{2\sqrt{-\mathbf{g}}}\frac{\partial\mathfrak{L}}{\partial(\partial_\iota g^{\mu\nu})}\partial_\varkappa g^{\mu\nu} + \frac{1}{\sqrt{-\mathbf{g}}}\frac{\partial}{\partial x^\rho}(\sqrt{-\mathbf{g}}\mathcal{S}^{\iota\rho}_\varkappa),\end{aligned} \tag{49}$$

and since $\sqrt{-\mathbf{g}}$ does not depend on the $\partial_\varkappa g^{\mu\nu}$ and $\partial_\rho\sqrt{-\mathbf{g}} = \Gamma^\sigma_{\rho\sigma}\sqrt{-\mathbf{g}}$ we can still write:

$$\begin{aligned}&R^\iota_\kappa - \frac{1}{2}\delta^\iota_\kappa R - \frac{1}{2}\delta^\iota_\kappa\Theta\\ &= -\frac{1}{2}\frac{\partial\Theta}{\partial(\partial_\iota g^{\mu\nu})}\partial_\varkappa g^{\mu\nu} + \frac{\partial}{\partial x^\rho}\mathcal{S}^{\iota\rho}_\varkappa\\ &\quad + \mathcal{S}^{\iota\rho}_\varkappa\Gamma^\sigma_{\rho\sigma},\end{aligned} \tag{50}$$

where $\mathcal{S}^{\nu\rho}_\mu$ is given by Eq.(113). Eq.(50) can now be compared with Eq.(3.10) of [30] (or with Eq.(1.4.10) in [29]) and we see that the last term, namely $\mathcal{S}^{\iota\rho}_\varkappa\Gamma^\sigma_{\rho\sigma}$ is missing there[17].

But leaving aside this "misprint", we then read, e.g., in [29] that:

"Therefore, the Freud identity requires two first order source tensors for the exterior gravitational problems in vacuum, as in Eq.(3.6.) of Ref.[1][18]. These terms are absent in Einstein's gravitation (1.4.1.)[19] that consequently, violates the Freud identity of Riemannian geometry".

First we must comment, that contrary to Santilli's statement, the two terms on the right member of Eq.(49) are *not* tensor fields, for indeed, from what has

[16] See page 70 of [17].
[17] However, the equation printed in [25] is correct.
[18] Ref. [1] is the reference [25] in the present paper.
[19] Eq.(1.4.1) in [29] is Einstein's field equation without source, i.e., $G_{\mu\nu} = 0$.

been said above and taking into account Eq.(37) we know that Freud's identity is simply the component version of a decomposition of the Einstein 3-form fields $\star\mathcal{G}^\mu$ in *two* parts (one of then an exact differential), which however are *not* indexed forms, and thus are gauge dependent objects. Second, it is necessary to become clear once and for ever that when $\star\mathbf{T}^\mu = 0$, we simply have $-\star\mathcal{G}^\mu = d\star\mathcal{S}^\mu + \star t^\mu = 0$., which is equivalent (Eq.(50) to:

$$\mathfrak{R}^\iota_\kappa - \frac{1}{2}\delta^\iota_\kappa\mathfrak{R} = -\mathfrak{U}^\iota_\kappa + \frac{1}{2}\left(\delta^\iota_\kappa\mathfrak{L} - \frac{\partial\mathfrak{L}}{\partial(\partial_\iota g^{\mu\nu})}\partial_\varkappa g^{\mu\nu}\right) = 0. \tag{51}$$

What can be inferred from this equation is simply that the Ricci tensor of the "exterior" problem is null.[20] And that is *all*, there is *no* inconsistency between Einstein gravity, the Einstein-Hilbert field equations and Freud's identity.

Remark 6. The fact that some people became confused during decades with Freud's identity and its real meaning [1, 29, 30, 40, 41, 42] may certainly be attributed to the use of the classical tensor calculus which, sometimes hides obvious things for a long time. That identity, contrary to the hopes of [1, 40, 41, 42] does *not* give a solution for the energy-momentum problem in GR, even if we introduce explicitly an energy-momentum tensor for the gravitational field, while maintaining that spacetime *is* a Lorentzian manifold. The root of the problem for non existence of an energy-momentum conservation law consists in the obvious fact that in GR there is not even sense to talk about the total energy momentum of particles following different worldlines. The reason is crystal clear: in any manifold not equipped with a teleparallel connection (as it is the case of a general Lorentzian manifold, with non zero curvature tensor), we cannot add vectors belonging to the tangent spaces at different spacetime points. The problem of finding an energy-momentum conservation law for matter fields in GR can be solved only in a few special cases, namely when there exists appropriate Killing vector fields in the Lorentzian manifold representing the gravitational field which the matters fields generate and where they live (see, details, e.g., in [20]).

Remark 7. We would also like to call the reader's attention to the fact that in [25] the quantity appearing in Definition II.11.3,

$$R^\iota_\kappa - \frac{1}{2}\delta^\iota_\kappa R - \frac{1}{2}\delta^\iota_\kappa\Theta, \tag{52}$$

is called the "completed Einstein tensor", and it is stated that its covariant derivative is null. This statement is wrong since the object given by Eq.(52) is not a tensor. Indeed although the two first terms define the Einstein tensor the term $\frac{1}{2}\delta^\iota_\kappa\Theta$ is *not* a a tensor field. We observe that already in 1916 Einstein at page 171 of the English translation of [6] explicitly said that Θ is an invariant *only* with respect to linear transformations of coordinates, i.e., it is not a scalar function in the manifold. Moreover, in a paper published in 1917 Levi-Civita, explicitly stated

[20]We are not going to discuss here if the exterior problem with a zero source term is a physically valid problem. We are convinced that it is not, but certainly Santilli's proposed solution for that problem inferred from his use of Freud's identity is not the answer to that important issue.

that Θ is *not* a scalar invariant [12] (see also [31])[21]. And, since $\frac{1}{2}\delta_\kappa^\iota\Theta$ is not a tensor field there is no meaning in taking its covariant derivative, and consequently Corollary II.11.2.1 in [25] is false.

5. Freud's Identity and the Energy-Momentum "Conservation Law" of GR

We already comment that Freud's identity through Eq.(32) (or Eq.(34)) suggests that we have found a conservation law for the energy-momentum of matter plus the gravitational field in GR. Indeed, from Eq.(34), it follows that

$$d(\star\mathbf{T}^\mu + \star t^\mu) = 0. \tag{53}$$

However, this is simply a wishful thinking, since the $\star t^\mu$ are gauge dependent quantities and that fact implies that one of the definitions of the "inertial" mass of the source, in GR given by [35]

$$m_{\mathbf{I}} = -\int_V \star\left(\mathbf{T}^0 + t^0\right) = \int_{\partial V} \star\,\mathcal{S}^0 \tag{54}$$

takes a value that depends on the coordinate system that we choose to make the computation.

In truth, Eq.(54), printed in many papers and books results from a naive use of Stokes theorem. Indeed, such a theorem is valid one for the integration of *true* differential forms (under well known conditions). If we recall the well known definition of the integral of a differential form [3, 7] we see that a coordinate free result depends fundamentally on the fact that the differential form being integrated defines a *true* tensor. However, as already mentioned in Remark **1**, the $\star\mathcal{S}^\mu$ are not true indexed forms, and so their integration will be certainly coordinate dependent [2]. In Appendix C for completeness and hoping that the present paper may be of some utility for people trying to understand this issue, we find also from our formalism the so called Einstein and the Landau-Lifshitz "inertial" masses (concepts which have the same problems as the one defined in Eq.(54)).

The problem just discussed is a really serious one if we take GR as a valid theory of the gravitational field, for it means that in that theory there are no conservation laws of energy-momentum (and also of angular momentum) despite almost 100 years of hard work by several people [22]. And, at this point it is better to quote page 98 of Sachs & Wu [22]:

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[21]By the way, a proof that Θ is not a scalar is as follows. Calculate its value at a given point spacetime point using arbitrary coordinates. You get in general that Θ is non null (you can verify this with an example, e.g., using the Schwarzschild in standard coordinates). Next introduce Riemann normal coordinates covering that spacetime point. Using these coordinates all connection are zero at that point and then the evaluation of Θ now gives zero.

[22]A detailed discussion of conservation laws in a general Riemann-Cartan spacetime is given [20].

" As mentioned in section 3.8, conservation laws have a great predictive power. It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity. Many of the attempts to resurrect it are quite interesting; many are simply garbage."

6. Conclusions

In this paper we proved that contrary to the claim in [29, 30], there is no incompatibility from the mathematical point of view between Freud's identity and Einstein-Hilbert field equations of GR, both in vacuum and inside matter. Freud's identity, or disguised versions of it, have been used by several people during all XXth century to try to give a meaning to conservation laws of energy-momentum and angular momentum in GR. These efforts unfortunately resulted in no success, of course, because Freud's identity involves the use of pseudo-tensors (something that is absolutely obvious in our presentation), and thus gives global quantities (i.e., the result of integrals) depending of the coordinate chart used (see also Appendices D and E). This is a serious and vexatious problem that we believe, will need a radical change of paradigm to be solved[23]. As discussed in, e.g., [16, 21] a possible solution (maintaining the Einstein-Hilbert equations in an appropriate form) can be given with the gravitational field interpreted as field in Faraday sense living in Minkowski spacetime (or other background spacetime equipped with absolute parallelism)[24]. The geometrical interpretation of gravitation as "geometry of spacetime" is a simple coincidence [16, 38], which we may bevalid only to a certain degree of approximation.

Appendix A. Clifford Bundle Formalism

Let $\mathfrak{M} = (M, \mathbf{g}, D, \tau_g, \uparrow)$ be an arbitrary Lorentzian spacetime. The quadruple $(M, \mathbf{g}, \tau_g, \uparrow)$ denotes a four-dimensional time-oriented and space-oriented Lorentzian manifold [21, 22]. This means that $\mathbf{g} \in \sec T_2^0 M$ is a Lorentzian metric of signature (1,3), $\tau_g \in \sec \bigwedge^4 T^*M$ and $\uparrow$ is a time-orientation (see details, e.g., in [22]). Here, T^*M $[TM]$ is the cotangent [tangent] bundle. $T^*M = \cup_{x\in M} T_x^* M$,

[23]Using the asymptotic flatness notions, first introduced by Penrose [18], it is possible, for some "isolated systems", to introduce the ADM and the Bondi masses. It is even possible to prove that the Bondi mass is positive [39]. But even if the notion of Bondi mass is considered by many a good solution to the energy-problem in GR, the fact is that it did *not* solve the problem in principle. It is only a calculational device. A good introduction to the notions asymptopita, ADM and Bondi masses can be found in [33].

[24]Recently Gorelik proposed in an interesting paper [9] to use the *quasi Poincaré group* of a Riemannian space as the generator of the Noether symmetries leading to conservation laws of "energy-momentum", angular momentum and "center of mass motion". A need comment on this approach that do not involve the use of the Freud's identity will be presented somewhere.

$TM = \cup_{x\in M} T_x M$, and $T_x M \simeq T_x^* M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space[25]. D is the Levi-Civita connection of $\mathbf{g}$, i.e., it is a metric compatible connection, which implies $D\mathbf{g} = 0$. In general, $\mathbf{R} = \mathbf{R}^D \neq 0$, $\Theta = \Theta^D = 0$, $\mathbf{R}$ and Θ being respectively the curvature and torsion tensors of the connection. Minkowski spacetime is the particular case of a Lorentzian spacetime for which $\mathbf{R} = 0$, $\Theta = 0$, and $M \simeq \mathbb{R}^4$. Let $\mathtt{g} \in \sec T_0^2 M$ be the metric of the *cotangent bundle*. The Clifford bundle of differential forms $\mathcal{C}\ell(M,\mathtt{g})$ is the bundle of algebras, i.e., $\mathcal{C}\ell(M,\mathtt{g}) = \cup_{x\in M}\mathcal{C}\ell(T_x^*M,\mathtt{g})$, where $\forall x \in M, \mathcal{C}\ell(T_x^*M,\mathtt{g}) = \mathbb{R}_{1,3}$, the so called *spacetime algebra* [21]. Recall also that $\mathcal{C}\ell(M,\mathtt{g})$ is a vector bundle associated to the *orthonormal frame bundle*, i.e., $\mathcal{C}\ell(M,\mathtt{g}) = P_{\mathrm{SO}^e_{(1,3)}}(M) \times_{\mathrm{Ad}} Cl_{1,3}$ [10, 15]. For any $x \in M$, $\mathcal{C}\ell(T_x^*M, \mathtt{g}|_x)$ as a linear space over the real field $\mathbb{R}$ is isomorphic to the Cartan algebra $\bigwedge T_x^*M$ of the cotangent space. $\bigwedge T_x^*M = \oplus_{k=0}^4 \bigwedge^k T_x^*M$, where $\bigwedge^k T_x^*M$ is the $\binom{4}{k}$-dimensional space of k-forms. Then, sections of $\mathcal{C}\ell(M,\mathtt{g})$ can be represented as a sum of non homogeneous differential forms, that will be called Clifford (multiform) fields. In the Clifford bundle formalism, of course, arbitrary basis can be used (see remark below), but in this short review of the main ideas of the Clifford calculus we use orthonormal basis. Let then $\{\mathbf{e_a}\}$ be an orthonormal basis for $TU \subset TM$, i.e., $\mathtt{g}(\mathbf{e_a},\mathbf{e_a}) = \eta_{\mathbf{ab}} = \mathrm{diag}(1,-1,-1,-1)$. Let $\boldsymbol{\theta}^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$ ($\mathbf{a} = 0,1,2,3$) be such that the set $\{\boldsymbol{\theta}^{\mathbf{a}}\}$ is the dual basis of $\{\mathbf{e_a}\}$.

A.1. Clifford Product

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by

$$\boldsymbol{\theta}^{\mathbf{a}}\boldsymbol{\theta}^{\mathbf{b}} + \boldsymbol{\theta}^{\mathbf{b}}\boldsymbol{\theta}^{\mathbf{a}} = 2\eta^{\mathbf{ab}} \tag{55}$$

and if $\mathcal{C} \in \sec \mathcal{C}\ell(M,\mathtt{g})$ we have

$$\mathcal{C} = s + v_{\mathbf{a}}^{\mathbf{a}}\boldsymbol{\theta} + \frac{1}{2!} f_{\mathbf{ab}}\boldsymbol{\theta}^{\mathbf{a}}\boldsymbol{\theta}^{\mathbf{b}} + \frac{1}{3!} t_{\mathbf{abc}}\boldsymbol{\theta}^{\mathbf{a}}\boldsymbol{\theta}^{\mathbf{b}}\boldsymbol{\theta}^{\mathbf{c}} + p\boldsymbol{\theta}^5 \ , \tag{56}$$

where $\tau_g = \boldsymbol{\theta}^5 = \boldsymbol{\theta}^0\boldsymbol{\theta}^{\mathbf{1}}\boldsymbol{\theta}^{\mathbf{2}}\boldsymbol{\theta}^{\mathbf{3}}$ is the volume element and s, $v_{\mathbf{a}}$, $f_{\mathbf{ab}}$, $t_{\mathbf{abc}}$, $p \in \sec \bigwedge^0 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$.

For $A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g}), B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$ we define the *exterior product* in $\mathcal{C}\ell(M,\mathtt{g})$ $(\forall r,s = 0,1,2,3)$ by

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}, \tag{57}$$

where $\langle\ \rangle_k$ is the component in $\bigwedge^k T^*M$ of the Clifford field. Of course, $A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r$, and the exterior product is extended by linearity to all sections of $\mathcal{C}\ell(M,\mathtt{g})$.

Let $A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g}), B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$. We define a *scalar product* in $\mathcal{C}\ell(M,\mathtt{g})$ (denoted by $\cdot$) as follows:

[25]Not to be confused with Minkowski spacetime [22].

(i) For $a, b \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$,

$$a \cdot b = \frac{1}{2}(ab + ba) = \mathtt{g}(a, b). \tag{58}$$

(ii) For $A_r = a_1 \wedge \ldots \wedge a_r, B_r = b_1 \wedge \ldots \wedge b_r$, $a_i, b_j \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, $i, j = 1, \ldots, r$,

$$\begin{aligned} A_r \cdot B_r &= (a_1 \wedge \ldots \wedge a_r) \cdot (b_1 \wedge \ldots \wedge b_r) \\ &= \begin{vmatrix} a_1 \cdot b_1 & \ldots & a_1 \cdot b_r \\ \ldots\ldots & \ldots & \ldots\ldots \\ a_r \cdot b_1 & \ldots & a_r \cdot b_r \end{vmatrix}. \end{aligned} \tag{59}$$

We agree that if $r = s = 0$, the scalar product is simply the ordinary product in the real field.

Also, if $r \neq s$, then $A_r \cdot B_s = 0$. Finally, the scalar product is extended by linearity for all sections of $\mathcal{C}\ell(M, \mathtt{g})$.

For $r \leq s$, $A_r = a_1 \wedge \ldots \wedge a_r$, $B_s = b_1 \wedge \ldots \wedge b_s$, we define the *left contraction* $\lrcorner : (A_r, B_s) \mapsto A_r \lrcorner B_s$ by

$$A_r \lrcorner B_s = \sum_{i_1 < \ldots < i_r} \epsilon^{i_1 \ldots i_s} (a_1 \wedge \ldots \wedge a_r) \cdot (b_{i_1} \wedge \ldots \wedge b_{i_r})^{\sim} b_{i_r+1} \wedge \ldots \wedge b_{i_s} \tag{60}$$

where $\sim$ is the reverse mapping (*reversion*) defined by

$$\begin{gathered} \tilde{} : \sec \mathcal{C}\ell(M, \mathtt{g}) \to \sec \mathcal{C}\ell(M, \mathtt{g}), \\ \tilde{A} = \sum_{p=0}^{4} \tilde{A}_p = \sum_{p=0}^{4} (-1)^{\frac{1}{2}k(k-1)} A_p, \\ A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g}). \end{gathered} \tag{61}$$

We agree that for $\alpha, \beta \in \sec \bigwedge^0 T^*M$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0 T^*M$, $A_r \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ then $(\alpha A_r) \lrcorner B_s = A_r \lrcorner (\alpha B_s)$. Left contraction is extended by linearity to all pairs of sections of $\mathcal{C}\ell(M, \mathtt{g})$, i.e., for $A, B \in \sec \mathcal{C}\ell(M, \mathtt{g})$

$$A \lrcorner B = \sum_{r,s} \langle A \rangle_r \lrcorner \langle B \rangle_s, \quad r \leq s. \tag{62}$$

It is also necessary to introduce the operator of *right contraction* denoted by $\llcorner$. The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $A_r \in \sec \bigwedge^r T^*M$, $B_s \in \sec \bigwedge^s T^*M$ then $A_r \llcorner (\alpha B_s) = (\alpha A_r) \llcorner B_s$. See also the third formula in Eq.(63).

The main formulas used in this paper can be obtained from the following ones

$$\begin{aligned}
a\mathcal{B}_s &= a\lrcorner\mathcal{B}_s + a\wedge\mathcal{B}_s,\ \ \mathcal{B}_s a = \mathcal{B}_s\llcorner a + \mathcal{B}_s\wedge a,\\
a\lrcorner\mathcal{B}_s &= \frac{1}{2}(a\mathcal{B}_s - (-1)^s\mathcal{B}_s a),\\
\mathcal{A}_r\lrcorner\mathcal{B}_s &= (-1)^{r(s-r)}\mathcal{B}_s\llcorner\mathcal{A}_r,\\
a\wedge\mathcal{B}_s &= \frac{1}{2}(a\mathcal{B}_s + (-1)^s\mathcal{B}_s a),\\
\mathcal{A}_r\mathcal{B}_s &= \langle\mathcal{A}_r\mathcal{B}_s\rangle_{|r-s|} + \langle\mathcal{A}_r\mathcal{B}_s\rangle_{|r-s|+2} + \ldots + \langle\mathcal{A}_r\mathcal{B}_s\rangle_{|r+s|}\\
&= \sum_{k=0}^{m}\langle\mathcal{A}_r\mathcal{B}_s\rangle_{|r-s|+2k}\\
\mathcal{A}_r\cdot\mathcal{B}_r &= \mathcal{B}_r\cdot\mathcal{A}_r = \widetilde{\mathcal{A}}_r\ \lrcorner\mathcal{B}_r = \mathcal{A}_r\llcorner\widetilde{\mathcal{B}}_r = \langle\widetilde{\mathcal{A}}_r\mathcal{B}_r\rangle_0 = \langle\mathcal{A}_r\widetilde{\mathcal{B}}_r\rangle_0.
\end{aligned} \tag{63}$$

Two other important identities to be used below are:

$$a\lrcorner(\mathcal{X}\wedge\mathcal{Y}) = (a\lrcorner\mathcal{X})\wedge\mathcal{Y} + \hat{\mathcal{X}}\wedge(a\lrcorner\mathcal{Y}), \tag{64}$$

for any $a\in\sec\bigwedge^1 T^*M$ and $\mathcal{X},\mathcal{Y}\in\sec\bigwedge T^*M$, and

$$A\lrcorner(B\lrcorner C) = (A\wedge B)\lrcorner C, \tag{65}$$

for any $A,B,C\in\sec\bigwedge T^*M\hookrightarrow\mathcal{C}\ell(M,\mathtt{g})$.

A.1.1. Hodge Star Operator. Let $\star$ be the Hodge star operator, i.e., the mapping

$$\star:\bigwedge^k T^*M\to\bigwedge^{4-k}T^*M,\ \ A_k\mapsto\star A_k \tag{66}$$

where for $A_k\in\sec\bigwedge^k T^*M\hookrightarrow\sec\mathcal{C}\ell(M,\mathtt{g})$

$$[B_k\cdot A_k]\tau_g = B_k\wedge\star A_k, \forall B_k\in\sec\bigwedge^k T^*M\hookrightarrow\sec\mathcal{C}\ell(M,\mathtt{g}). \tag{67}$$

$\tau_{\mathtt{g}} = \theta^{\mathbf{5}}\in\sec\bigwedge^4 T^*M\hookrightarrow\sec\mathcal{C}\ell(M,\mathtt{g})$ is a *standard* volume element. Then we can easily verify that

$$\star A_k = \widetilde{A}_k\tau_{\mathtt{g}} = \widetilde{A}_k\lrcorner\tau_{\mathtt{g}}. \tag{68}$$

where as noted before, in this paper $\widetilde{\mathcal{A}}_k$ denotes the *reverse* of $\mathcal{A}_k$. Eq.(68) permits calculation of Hodge duals very easily in an orthonormal basis for which $\tau_{\mathtt{g}} = \boldsymbol{\theta}^{\mathbf{5}}$. Let $\{\vartheta^\alpha\}$ be the dual basis of $\{e_\alpha\}$ (i.e., it is a basis for $T^*U\equiv\bigwedge^1 T^*U$) which is either orthonormal or a coordinate basis. Then writing $\mathtt{g}(\vartheta^\alpha,\vartheta^\beta) = g^{\alpha\beta}$, with $g^{\alpha\beta}g_{\alpha\rho} = \delta^\beta_\rho$, and $\vartheta^{\mu_1\ldots\mu_p} = \vartheta^{\mu_1}\wedge\ldots\wedge\vartheta^{\mu_p}$, $\vartheta^{\nu_{p+1}\ldots\nu_n} = \vartheta^{\nu_{p+1}}\wedge\ldots\wedge\vartheta^{\nu_n}$ we have from Eq.(68)

$$\star\,\theta^{\mu_1\ldots\mu_p} = \frac{1}{(n-p)!}\sqrt{|\mathbf{g}|}g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\epsilon_{\nu_1\ldots\nu_n}\vartheta^{\nu_{p+1}\ldots\nu_n}. \tag{69}$$

where $\mathbf{g}$ denotes the determinant of the matrix with entries $g_{\alpha\beta} = \mathtt{g}(e_\alpha,e_\beta)$, i.e.,

$$\mathbf{g} = \det[g_{\alpha\beta}]. \tag{70}$$

We also define the inverse $\star^{-1}$ of the Hodge dual operator, such that $\star^{-1}\star = \star\star^{-1} = 1$. It is given by:

$$\begin{aligned}&\star^{-1} : \sec \bigwedge\nolimits^{n-r} T^*M \to \sec \bigwedge\nolimits^{r} T^*M,\\ &\star^{-1} = (-1)^{r(n-r)} \text{sgn } \mathbf{g} \star,\end{aligned} \tag{71}$$

where sgn $\mathbf{g} = \mathbf{g}/|\mathbf{g}|$ denotes the sign of the determinant $\mathbf{g}$.

Some useful identities (used several times below) involving the Hodge star operator, the exterior product and contractions are:

$$\begin{aligned}&A_r \wedge \star B_s = B_s \wedge \star A_r; \quad r = s\\ &A_r \cdot \star B_s = B_s \cdot \star A_r; \quad r + s = n\\ &A_r \wedge \star B_s = (-1)^{r(s-1)} \star (\tilde{A}_r \lrcorner B_s); \quad r \leq s\\ &A_r \lrcorner \star B_s = (-1)^{rs} \star (\tilde{A}_r \wedge B_s); \quad r + s \leq n\\ &\star \tau_g = \text{sign } \mathbf{g}; \quad \star 1 = \tau_g.\end{aligned} \tag{72}$$

A.1.2. Dirac Operator Associated to a Levi-Civita Connection. Let d and δ be respectively the differential and Hodge codifferential operators acting on sections of $\mathcal{C}\ell(M, \mathtt{g})$. If $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, then $\delta A_p = (-1)^p \star^{-1} d \star A_p$.

The Dirac operator acting on sections of $\mathcal{C}\ell(M, \mathtt{g})$ associated with the metric compatible connection D is the invariant first order differential operator

$$\boldsymbol{\partial} = \boldsymbol{\theta}^{\mathbf{a}} D_{\mathbf{e_a}}, \tag{73}$$

where $\{\mathbf{e_a}\}$ is an arbitrary *orthonormal basis* for $TU \subset TM$ and $\{\boldsymbol{\theta}^{\mathbf{b}}\}$ is a basis for $T^*U \subset T^*M$ dual to the basis $\{\mathbf{e_a}\}$, i.e., $\boldsymbol{\theta}^{\mathbf{b}}(\mathbf{e_a}) = \delta^{\mathbf{a}}_{\mathbf{b}}$, $\mathbf{a}, \mathbf{b} = 0, 1, 2, 3$. The reciprocal basis of $\{\boldsymbol{\theta}^{\mathbf{b}}\}$ is denoted $\{\theta_{\mathbf{a}}\}$ and we have $\theta_{\mathbf{a}} \cdot \theta_{\mathbf{b}} = \eta_{\mathbf{ab}}$. Also,

$$D_{\mathbf{e_a}} \boldsymbol{\theta}^{\mathbf{b}} = -\omega^{\mathbf{bc}}_{\mathbf{a}} \boldsymbol{\theta}_{\mathbf{c}} \tag{74}$$

and we write the connection 1-forms in the orthogonal gauge as

$$\omega^{\mathbf{a}}_{\mathbf{b}} := \omega^{\mathbf{a}}_{\mathbf{cb}} \boldsymbol{\theta}^{\mathbf{c}}. \tag{75}$$

Moreover, we introduce the objects $\omega_{\mathbf{e_a}} \in \sec \bigwedge^2 T^*M$,

$$\omega_{\mathbf{e_a}} = \frac{1}{2} \omega^{\mathbf{bc}}_{\mathbf{a}} \boldsymbol{\theta}_{\mathbf{b}} \wedge \boldsymbol{\theta}_{\mathbf{c}}. \tag{76}$$

Then, for any $A_p \in \sec \bigwedge^p T^*M$, $p = 0, 1, 2, 3, 4$ we can write

$$D_{\mathbf{e_a}} A_p = \partial_{\mathbf{e_a}} A_p + \frac{1}{2} [\omega_{\mathbf{e_a}}, A_p], \tag{77}$$

where $\partial_{\mathbf{e_a}}$ is the Pfaff derivative, i.e., if $A_p = \frac{1}{p!} A_{\mathbf{i}_1 \ldots \mathbf{i}_p} \theta^{\mathbf{i}_1 \ldots \mathbf{i}_p}$,

$$\partial_{\mathbf{e_a}} A_p := \frac{1}{p!} \mathbf{e_a}(A_{\mathbf{i}_1 \ldots \mathbf{i}_p}) \boldsymbol{\theta}^{\mathbf{i}_1 \ldots \mathbf{i}_p}. \tag{78}$$

Eq.(77) is an important formula which is also valid for a nonhomogeneous $A \in \sec \mathcal{C}\ell(M, \mathtt{g})$. It is proved, e.g., in [15, 21].

We have also the important result:

$$\begin{aligned}\boldsymbol{\partial} A_p &= \boldsymbol{\partial} \wedge A_p + \boldsymbol{\partial} \lrcorner A_p = dA_p - \delta A_p,\\ \boldsymbol{\partial} \wedge A_p &= dA_p, \quad \boldsymbol{\partial} \lrcorner A_p = -\delta A_p.\end{aligned} \tag{79}$$

Remark 8. We conclude this section by emphasizing that the formalism just presented is valid in an arbitrary coordinate basis $\{\boldsymbol{\partial}_\mu\}$ of $TU \subset TM$ associated to local coordinates $\{x^\mu\}$ covering U. In this case if $\{\boldsymbol{\theta}^\mu = dx^\mu\}$ is the dual basis of $\{\boldsymbol{\partial}_\mu\}$ we write

$$D_{\partial_\mu} \partial_\nu = \Gamma^\rho_{\mu\nu} \partial_\rho \qquad D_{\partial_\mu} \boldsymbol{\gamma}^\beta = -\Gamma^\beta_{\mu\alpha} \boldsymbol{\gamma}^\alpha. \tag{80}$$

We also write the connection 1-forms in a coordinate gauge as:

$$\boldsymbol{\Gamma}^\alpha_\beta := \Gamma^\alpha_{\mu\beta} \boldsymbol{\theta}^\mu. \tag{81}$$

A.2. Algebraic Derivatives of Functionals

Let $X \in \sec \bigwedge^p T^*M$. A functional F is a mapping

$$F : \sec \bigwedge^p T^*M \to \sec \bigwedge^r T^*M.$$

When no confusion arises we use a sloppy notation and denote the image $F(X) \in \sec \bigwedge^r T^*M$ simply by F, or vice versa. Which object we are talking about is always obvious from the context of the equations where they appear.

Let also $\boldsymbol{\delta} X \in \sec \bigwedge^p T^*M$. We define the variation of F as the functional $\boldsymbol{\delta} F \in \sec \bigwedge^r T^*M$ given by

$$\boldsymbol{\delta} F = \lim_{\lambda \to 0} \frac{F(X + \lambda \boldsymbol{\delta} X) - F(X)}{\lambda}. \tag{82}$$

Moreover, we define the algebraic derivative of $F(X)$ relative to X , denoted $\frac{\partial F}{\partial X}$ by:

$$\boldsymbol{\delta} F = \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X}. \tag{83}$$

Moreover, given $F : \sec \bigwedge^p T^*M \to \sec \bigwedge^r T^*M$, $G : \sec \bigwedge^p T^*M \to \sec \bigwedge^s T^*M$ the variation $\boldsymbol{\delta}$ satisfies

$$\boldsymbol{\delta}(F \wedge G) = \boldsymbol{\delta} F \wedge G + F \wedge \boldsymbol{\delta} G, \tag{84}$$

and the algebraic derivative satisfies (as it is trivial to verify)

$$\frac{\partial}{\partial X}(F \wedge G) = \frac{\partial F}{\partial X} \wedge G + (-1)^{rp} F \wedge \frac{\partial G}{\partial X}. \tag{85}$$

An important property of $\boldsymbol{\delta}$ is that it commutes with the exterior derivative operator d, i.e., for any given functional F

$$d\boldsymbol{\delta} F = \boldsymbol{\delta} dF. \tag{86}$$

In general we may have functionals depending on several different forms fields, say, $F(X,Y) \in \sec \bigwedge^r T^*M$, and $X \in \sec \bigwedge^p T^*M$, $Y \in \sec \bigwedge^q T^*M$. In this case we have (using sloop notation):

$$\boldsymbol{\delta} F = \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} + \boldsymbol{\delta} Y \wedge \frac{\partial F}{\partial Y}. \tag{87}$$

An important case happens for a functional F such that $F(X,dX) \in \sec \bigwedge^n T^*M$ where $n = \dim M$ is the manifold dimension. In this case, for $U \subset M$, we can write supposing that the variation $\boldsymbol{\delta} X$ is chosen to be null in the boundary ∂U (or that $\frac{\partial F}{\partial dX}\big|_{\partial U} = 0$) and taking into account Stokes theorem,

$$\begin{aligned}
\boldsymbol{\delta} \int_U F &:= \int_U \boldsymbol{\delta} F = \int_U \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} + \boldsymbol{\delta} dX \wedge \frac{\partial F}{\partial dX} \\
&= \int_U \boldsymbol{\delta} X \wedge \left[\frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right)\right] + d\left(\boldsymbol{\delta} X \wedge \frac{\partial F}{\partial dX}\right) \\
&= \int_U \boldsymbol{\delta} X \wedge \left[\frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right)\right] + \int_{\partial U} \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial dX} \\
&= \int_U \boldsymbol{\delta} X \wedge \frac{\boldsymbol{\delta} F}{\boldsymbol{\delta} X},
\end{aligned} \tag{88}$$

where $\frac{\boldsymbol{\delta}}{\boldsymbol{\delta} X} F(X,dX) \in \sec \bigwedge^{n-p} T^*M$ is called the *functional derivative* of F and we have:

$$\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta} X} = \frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right). \tag{89}$$

When $F = \mathcal{L}$ is a Lagrangian density in field theory $\frac{\boldsymbol{\delta} \mathcal{L}}{\boldsymbol{\delta} X}$ is called the Euler-Lagrange functional.[26]

Appendix B. Variation of the Einstein-Hilbert Lagrangian Density $\mathcal{L}_{EH}$

We have from $\mathcal{L}_{EH} = \frac{1}{2}\mathcal{R}_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})$,

$$\begin{aligned}
\boldsymbol{\delta} \mathfrak{L}_{EH} &= \frac{1}{2}\boldsymbol{\delta}[\mathcal{R}_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] \\
&= \frac{1}{2}\boldsymbol{\delta}\mathcal{R}_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) + \frac{1}{2}\mathcal{R}_{\mathbf{cd}} \wedge \boldsymbol{\delta} \star (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}).
\end{aligned} \tag{90}$$

[26] A detailed theory of derivatives of non homogeneous multiform functions of multiple non homogeneous multiform variables may be found in [21].

From Cartan's second structure equation we can write

$$
\begin{aligned}
&\boldsymbol{\delta}\mathcal{R}_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&= \boldsymbol{\delta} d\omega_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) + \boldsymbol{\delta}\omega_{\mathbf{ck}} \wedge \omega_{\mathbf{d}}^{\mathbf{k}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) + \omega_{\mathbf{ck}} \wedge \boldsymbol{\delta}\omega_{\mathbf{d}}^{\mathbf{k}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&= \boldsymbol{\delta} d\omega_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \qquad (91) \\
&= d[\boldsymbol{\delta}\omega_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] - \boldsymbol{\delta}\omega_{\mathbf{cd}} \wedge d[\star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})]. \\
&= d[\boldsymbol{\delta}\omega_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] - \boldsymbol{\delta}\omega_{\mathbf{cd}} \wedge [-\omega_{\mathbf{k}}^{\mathbf{c}} \wedge \star(\boldsymbol{\theta}^{\mathbf{k}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) - \omega_{\mathbf{k}}^{\mathbf{d}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{k}})] \\
&= d[\boldsymbol{\delta}\omega_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})].
\end{aligned}
$$

Moreover, using the definition of algebraic derivative (Eq.(82)) we have

$$
\boldsymbol{\delta} \star (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) := \boldsymbol{\delta\theta}^{\mathbf{m}} \wedge \frac{\partial[\star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})}{\partial \boldsymbol{\theta}^{\mathbf{m}}} \tag{92}
$$

Now recalling Eq.(69) of Appendix A we can write

$$
\begin{aligned}
\boldsymbol{\delta} \star (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) &= \boldsymbol{\delta}(\frac{1}{2}\eta^{\mathbf{ck}}\eta^{\mathbf{dl}}\epsilon_{\mathbf{klmn}}\boldsymbol{\theta}^{\mathbf{m}} \wedge \boldsymbol{\theta}^{\mathbf{n}}) \\
&= \boldsymbol{\delta\theta}^{\mathbf{m}} \wedge (\eta^{\mathbf{ck}}\eta^{\mathbf{dl}}\epsilon_{\mathbf{klmn}}\boldsymbol{\theta}^{\mathbf{n}}),
\end{aligned} \tag{93}
$$

from where we get

$$
\frac{\partial \star (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})}{\partial \boldsymbol{\theta}^{\mathbf{m}}} = \eta^{\mathbf{ck}}\eta^{\mathbf{dl}}\epsilon_{\mathbf{klmn}}\boldsymbol{\theta}^{\mathbf{n}}. \tag{94}
$$

On the other hand we have recalling Eq.(60) of Appendix A

$$
\begin{aligned}
&\boldsymbol{\theta}_{\mathbf{m}} \lrcorner \star (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) = \boldsymbol{\theta}_{\mathbf{m}} \lrcorner (\frac{1}{2}\eta^{\mathbf{ck}}\eta^{\mathbf{dl}}\epsilon_{\mathbf{klrs}}\boldsymbol{\theta}^{\mathbf{r}} \wedge \boldsymbol{\theta}^{\mathbf{s}}) \\
&= \eta^{\mathbf{ck}}\eta^{\mathbf{dl}}\epsilon_{\mathbf{klmn}}\boldsymbol{\theta}^{\mathbf{n}}.
\end{aligned} \tag{95}
$$

Moreover, using the fourth formula in Eq.(72) of Appendix A, we can write

$$
\begin{aligned}
\frac{\partial[\star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})}{\partial \boldsymbol{\theta}^{\mathbf{m}}} &= \boldsymbol{\theta}_{\mathbf{m}} \lrcorner \star (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) \\
&= \star[\boldsymbol{\theta}_{\mathbf{m}} \wedge (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] = \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}} \wedge \boldsymbol{\theta}_{\mathbf{m}}).
\end{aligned} \tag{96}
$$

Finally,

$$
\boldsymbol{\delta} \star (\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}}) = \boldsymbol{\delta\theta}^{\mathbf{m}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}} \wedge \boldsymbol{\theta}_{\mathbf{m}}). \tag{97}
$$

Then using Eq.(91) and Eq.(97) in Eq.(90) we get

$$
\boldsymbol{\delta}\mathfrak{L}_{EH} = \frac{1}{2}d[\boldsymbol{\delta}\omega_{\mathbf{cd}} \wedge \star(\boldsymbol{\theta}^{\mathbf{c}} \wedge \boldsymbol{\theta}^{\mathbf{d}})] + \boldsymbol{\delta\theta}^{\mathbf{m}} \wedge [\frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \star(\boldsymbol{\theta}^{\mathbf{a}} \wedge \boldsymbol{\theta}^{\mathbf{b}} \wedge \boldsymbol{\theta}_{\mathbf{m}})]. \tag{98}
$$

Now,

$$\begin{aligned}\frac{1}{2}\mathcal{R}_{\mathbf{ab}}\wedge\star(\theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\wedge\theta_{\mathbf{m}}) &= -\frac{1}{2}\star[\mathcal{R}_{\mathbf{ab}}\lrcorner(\boldsymbol{\theta}^{\mathbf{a}}\wedge\boldsymbol{\theta}^{\mathbf{b}}\wedge\boldsymbol{\theta}_{\mathbf{m}})]\\ &= -\frac{1}{4}R_{\mathbf{abck}}\star[(\boldsymbol{\theta}^{\mathbf{c}}\wedge\boldsymbol{\theta}^{\mathbf{k}})\lrcorner(\boldsymbol{\theta}^{\mathbf{a}}\wedge\boldsymbol{\theta}^{\mathbf{b}}\wedge\boldsymbol{\theta}_{\mathbf{m}})]\\ &= -\star(\mathcal{R}_{\mathbf{m}}-\frac{1}{2}R\boldsymbol{\theta}_{\mathbf{m}}) = -\star\mathcal{G}_{\mathbf{m}}, \end{aligned} \tag{99}$$

and so we can write

$$\int \boldsymbol{\delta}(\mathfrak{L}_{EH}+\mathfrak{L}_m) = \int \boldsymbol{\delta\theta}^{\mathbf{a}}\wedge(-\star\mathcal{G}_{\mathbf{a}}+\frac{\partial\mathfrak{L}_m}{\partial\boldsymbol{\theta}^{\mathbf{a}}}) = 0. \tag{100}$$

Appendix C. Calculation of the Components of $\mathcal{S}_\lambda$

Here, using the powerful Clifford bundle formalism we present two calculations[27] of the components of $\mathcal{S}_\lambda$ given by Eq.(35) in a *coordinate basis. We* directly identify Freud's objects $\mathfrak{U}_\mu^{\lambda\sigma}$ and Freud's identity as given in [8]. We start from

$$\star\mathcal{S}_\lambda = \frac{1}{2}\boldsymbol{\Gamma}_{\alpha\beta}\wedge\star(\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda). \tag{101}$$

Using the third formula in Eq.(72) of Appendix A we can write

$$\star\mathcal{S}_\lambda = \boldsymbol{\Gamma}_{\alpha\beta}\wedge\star(\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda) = \star\left[\frac{1}{2}\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner(\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda)\right] \tag{102}$$

or

$$\mathcal{S}_\lambda = \frac{1}{2}\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner(\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda) \tag{103}$$

Using now Eq.(60) of Appendix A we have

$$\mathcal{S}_\lambda = \frac{1}{2}\left\{(\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner\boldsymbol{\gamma}^\alpha)\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda - (\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner\boldsymbol{\gamma}^\beta)\wedge\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}_\lambda + (\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner\boldsymbol{\gamma}_\lambda)\wedge\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}^\beta\right\} \tag{104}$$

Now,

$$\begin{aligned}(\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner\boldsymbol{\gamma}^\alpha)\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda &= (\boldsymbol{\gamma}^\alpha\lrcorner\boldsymbol{\Gamma}_{\alpha\beta})\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda\\ &\overset{\text{Eq.(64)}}{=} \boldsymbol{\gamma}_\alpha\lrcorner(\boldsymbol{\Gamma}_\beta^\alpha\wedge\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda)+\boldsymbol{\Gamma}_\beta^\alpha\wedge(\boldsymbol{\gamma}_\alpha\lrcorner(\boldsymbol{\gamma}^\beta\wedge\boldsymbol{\gamma}_\lambda))\\ &\overset{\text{Eq.(18)}}{=} -\boldsymbol{\gamma}_\alpha\lrcorner(d\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}_\lambda)+\boldsymbol{\Gamma}_\beta^\alpha\wedge(\delta_\alpha^\beta\boldsymbol{\gamma}_\mu - g_{\alpha\lambda}\boldsymbol{\gamma}^\beta)\\ &\overset{\text{Eq.(64)}}{=} -(\boldsymbol{\gamma}_\alpha\lrcorner d\boldsymbol{\gamma}^\alpha)\wedge\boldsymbol{\gamma}_\lambda - d\boldsymbol{\gamma}^\alpha\wedge(\boldsymbol{\gamma}_\alpha\lrcorner\boldsymbol{\gamma}_\lambda)\\ &\qquad + \boldsymbol{\Gamma}_\alpha^\alpha\wedge\boldsymbol{\gamma}_\mu - g_{\alpha\lambda}\boldsymbol{\Gamma}_\beta^\alpha\wedge\boldsymbol{\gamma}^\beta\\ &= -(\boldsymbol{\gamma}_\alpha\lrcorner d\boldsymbol{\gamma}^\alpha)\wedge\boldsymbol{\gamma}_\lambda - g_{\alpha\lambda}(d\boldsymbol{\gamma}^\alpha+\boldsymbol{\Gamma}_\beta^\alpha\wedge\boldsymbol{\gamma}^\beta)+\boldsymbol{\Gamma}_\alpha^\alpha\wedge\boldsymbol{\gamma}_\mu\\ &\overset{\text{Eq.(18)}}{=} -(\boldsymbol{\gamma}_\alpha\lrcorner d\boldsymbol{\gamma}^\alpha)\wedge\boldsymbol{\gamma}_\lambda+\boldsymbol{\Gamma}_\alpha^\alpha\wedge\boldsymbol{\gamma}_\mu. \end{aligned} \tag{105}$$

[27]The second one is close to the one given in [35].

Analogously we find

$$\begin{aligned}(\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner\boldsymbol{\gamma}^{\beta})\wedge\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}_{\lambda}&=(\boldsymbol{\gamma}^{\alpha}\lrcorner d\boldsymbol{\gamma}_{\alpha})\wedge\boldsymbol{\gamma}_{\lambda}+\boldsymbol{\Gamma}_{\alpha}^{\alpha}\wedge\boldsymbol{\gamma}_{\lambda},\\ (\boldsymbol{\Gamma}_{\alpha\beta}\lrcorner\boldsymbol{\gamma}_{\lambda})\wedge\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}^{\beta}&=(\boldsymbol{\gamma}_{\lambda}\lrcorner d\boldsymbol{\gamma}^{\alpha})\wedge\boldsymbol{\gamma}_{\alpha}-d\boldsymbol{\gamma}_{\lambda},\end{aligned}\tag{106}$$

from where we can write

$$\mathcal{S}_{\mu}=\frac{1}{2}\left[-(\boldsymbol{\gamma}_{\alpha}\lrcorner d\boldsymbol{\gamma}^{\alpha})\wedge\boldsymbol{\gamma}_{\mu}-(\boldsymbol{\gamma}^{\alpha}\lrcorner d\boldsymbol{\gamma}_{\alpha})\wedge\boldsymbol{\gamma}_{\mu}+(\boldsymbol{\gamma}_{\mu}\lrcorner d\boldsymbol{\gamma}^{\alpha})\wedge\boldsymbol{\gamma}_{\alpha}-d\boldsymbol{\gamma}_{\mu}\right],\tag{107}$$

which taking account that $d\boldsymbol{\gamma}^{\alpha}=d^{2}x^{\alpha}=0$, reduces to

$$\mathcal{S}_{\mu}=-\frac{1}{2}\left[(\boldsymbol{\gamma}^{\alpha}\lrcorner d\boldsymbol{\gamma}_{\alpha})\wedge\boldsymbol{\gamma}_{\mu}+d\boldsymbol{\gamma}_{\mu}\right].\tag{108}$$

Now from Eq.(79) valid for a Levi-Civita connection for any $A\in\sec T^{*}M\hookrightarrow\mathcal{C}\ell(M,\mathtt{g})$ it is $dA=\boldsymbol{\partial}\wedge A$. So, we can write (recalling that $D_{\kappa}g_{\lambda\rho}=0$):

$$\begin{aligned}d\boldsymbol{\gamma}_{\mu}&=\boldsymbol{\gamma}^{\kappa}\wedge D_{\partial_{\kappa}}(g_{\mu\rho}\boldsymbol{\gamma}^{\rho})\\ &=(\partial_{\kappa}g_{\mu\rho}-g_{\mu\beta}\Gamma_{\kappa\rho}^{\beta})\boldsymbol{\gamma}^{\kappa}\wedge\boldsymbol{\gamma}^{\rho}\\ &=g_{\beta\rho}\Gamma_{\mu\kappa}^{\beta}\boldsymbol{\gamma}^{\kappa}\wedge\boldsymbol{\gamma}^{\rho}\\ &=\delta_{\beta}^{\sigma}\Gamma_{\mu\rho}^{\beta}g^{\kappa\lambda}\boldsymbol{\gamma}_{\lambda}\wedge\boldsymbol{\gamma}_{\sigma}\\ &=\frac{1}{2}\left(\delta_{\beta}^{\sigma}\Gamma_{\mu\rho}^{\beta}g^{\kappa\lambda}-\delta_{\beta}^{\lambda}\Gamma_{\mu\rho}^{\beta}g^{\kappa\sigma}\right)\boldsymbol{\gamma}_{\lambda}\wedge\boldsymbol{\gamma}_{\sigma}.\end{aligned}\tag{109}$$

Also,

$$\begin{aligned}\boldsymbol{\gamma}^{\alpha}\lrcorner d\boldsymbol{\gamma}_{\alpha}&=\boldsymbol{\gamma}^{\alpha}\lrcorner(g_{\beta\rho}\Gamma_{\alpha\kappa}^{\beta}\boldsymbol{\gamma}^{\kappa}\wedge\boldsymbol{\gamma}^{\rho})\\ &=g^{\alpha\kappa}g_{\beta\rho}\Gamma_{\alpha\kappa}^{\beta}\boldsymbol{\gamma}^{\rho}-g^{\alpha\rho}g_{\beta\rho}\Gamma_{\alpha\kappa}^{\beta}\boldsymbol{\gamma}^{\kappa}\\ &=(g^{\alpha\kappa}g_{\beta\rho}\Gamma_{\alpha\kappa}^{\beta}-\Gamma_{\alpha\kappa}^{\alpha})\boldsymbol{\gamma}^{\rho},\end{aligned}\tag{110}$$

and then

$$\begin{aligned}&(\boldsymbol{\gamma}^{\alpha}\lrcorner d\boldsymbol{\gamma}_{\alpha})\wedge\boldsymbol{\gamma}_{\mu}\\ &=(\delta_{\mu}^{\sigma}g^{\alpha\kappa}\Gamma_{\alpha\kappa}^{\lambda}-\delta_{\mu}^{\sigma}g^{\rho\lambda}\Gamma_{\alpha\rho}^{\alpha})\boldsymbol{\gamma}_{\lambda}\wedge\boldsymbol{\gamma}_{\sigma}.\end{aligned}\tag{111}$$

So, we get

$$\begin{aligned}\mathcal{S}_{\mu}&=-\frac{1}{2}\left(\delta_{\beta}^{\sigma}\Gamma_{\mu\rho}^{\beta}g^{\kappa\lambda}+\delta_{\mu}^{\sigma}g^{\alpha\kappa}\Gamma_{\alpha\kappa}^{\lambda}-\delta_{\mu}^{\sigma}g^{\rho\lambda}\Gamma_{\alpha\rho}^{\alpha}\right)\boldsymbol{\gamma}_{\lambda}\wedge\boldsymbol{\gamma}_{\sigma}\\ &=\frac{1}{2}\left\{\frac{1}{2}\det\begin{bmatrix}\delta_{\mu}^{\lambda}&\delta_{\mu}^{\sigma}&\delta_{\mu}^{\iota}\\ g^{\lambda\kappa}&g^{\sigma\kappa}&g^{\iota\kappa}\\ \Gamma_{\kappa\iota}^{\lambda}&\Gamma_{\kappa\iota}^{\sigma}&\Gamma_{\iota\kappa}^{\iota}\end{bmatrix}\right\}\boldsymbol{\gamma}_{\lambda}\wedge\boldsymbol{\gamma}_{\sigma}\\ &=\frac{1}{2}\mathcal{S}_{\mu}^{\lambda\sigma}\boldsymbol{\gamma}_{\lambda}\wedge\boldsymbol{\gamma}_{\sigma},\end{aligned}\tag{112}$$

and then

$$\mathcal{S}_{\mu}^{\lambda\sigma}=\frac{1}{2}\det\begin{bmatrix}\delta_{\mu}^{\lambda}&\delta_{\mu}^{\sigma}&\delta_{\mu}^{\iota}\\ g^{\lambda\kappa}&g^{\sigma\kappa}&g^{\iota\kappa}\\ \Gamma_{\kappa\iota}^{\lambda}&\Gamma_{\kappa\iota}^{\sigma}&\Gamma_{\iota\kappa}^{\iota}\end{bmatrix}.\tag{113}$$

C.1. Freud's $\mathfrak{U}_{\mu}^{\lambda\sigma}$

Now putting

$$\mathfrak{g}^{\sigma\nu} = \sqrt{-\mathbf{g}}g^{\sigma\nu},\ \mathfrak{g}_{\lambda\sigma} = \frac{1}{\sqrt{-\mathbf{g}}}g_{\lambda\sigma} \tag{114}$$

we recognize looking at the last formula in Freud's paper [8] that his $\mathfrak{U}_{\mu}^{\lambda\sigma}$ is given by

$$\mathfrak{U}_{\mu}^{\lambda\sigma} = \sqrt{-\mathbf{g}}\mathcal{S}_{\mu}^{\lambda\sigma}. \tag{115}$$

C.2. An Equivalent Formula for Freud's $\mathfrak{U}_{\mu}^{\lambda\sigma}$

We start again our computation of $\mathfrak{U}_{\mu}^{\lambda\sigma}$, recalling that from (Eq.(79)) we have for the Hodge coderivative

$$\begin{aligned}\delta\boldsymbol{\gamma}^{\alpha} &= -\boldsymbol{\partial}\lrcorner\boldsymbol{\gamma}^{\alpha} = -\boldsymbol{\gamma}^{\kappa}\lrcorner(D_{\partial_{\kappa}}\boldsymbol{\gamma}^{\alpha}) \\ &= \boldsymbol{\gamma}^{\kappa}\lrcorner(\Gamma_{\kappa\rho}^{\alpha}\boldsymbol{\gamma}^{\rho}) = g^{\kappa\rho}\Gamma_{\kappa\rho}^{\alpha},\end{aligned} \tag{116}$$

and then

$$\begin{aligned}\boldsymbol{\gamma}^{\alpha}\lrcorner d\boldsymbol{\gamma}_{\alpha} &= -2\boldsymbol{\Gamma}_{\alpha}^{\alpha} + (\boldsymbol{\gamma}^{\alpha}\lrcorner\boldsymbol{\Gamma}_{\beta\alpha})\boldsymbol{\gamma}^{\beta} + \boldsymbol{\gamma}^{\alpha}\lrcorner\boldsymbol{\Gamma}_{\alpha\beta})\boldsymbol{\gamma}^{\beta} \\ &= -2\boldsymbol{\Gamma}_{\alpha}^{\alpha} + \boldsymbol{\Gamma}_{\alpha}^{\alpha} + \boldsymbol{\gamma}_{\alpha}\delta\boldsymbol{\gamma}^{\alpha} \\ &= -\boldsymbol{\Gamma}_{\alpha}^{\alpha} + \boldsymbol{\gamma}_{\alpha}\delta\boldsymbol{\gamma}^{\alpha}.\end{aligned} \tag{117}$$

Using this result in Eq.(108) we get

$$\mathcal{S}_{\lambda} = -\frac{1}{2}\left(-\Gamma_{\alpha}^{\alpha}\wedge\boldsymbol{\gamma}_{\lambda} + (\boldsymbol{\gamma}_{\alpha}\wedge\boldsymbol{\gamma}_{\lambda})\delta\boldsymbol{\gamma}^{\alpha} + d\boldsymbol{\gamma}_{\lambda}\right). \tag{118}$$

Recalling that $\mathbf{g} = \det[g_{\alpha\beta}]$ we have the well known result [11]

$$d\mathbf{g} = (\partial_{\alpha}\mathbf{g})\boldsymbol{\gamma}^{\alpha} = 2\mathbf{g}\Gamma_{\alpha\kappa}^{\kappa}\boldsymbol{\gamma}^{\alpha} = 2\mathbf{g}\boldsymbol{\Gamma}_{\kappa}^{\kappa}, \tag{119}$$

and we can write

$$\begin{aligned}\mathcal{S}_{\lambda} &= -\frac{1}{2}\left(-\frac{d\mathbf{g}}{\mathbf{g}}\wedge\boldsymbol{\gamma}_{\lambda} + (\boldsymbol{\gamma}_{\alpha}\wedge\boldsymbol{\gamma}_{\lambda})\delta\boldsymbol{\gamma}^{\alpha} + d\boldsymbol{\gamma}_{\lambda} + \frac{1}{2}\frac{d\mathbf{g}}{\mathbf{g}}\wedge\boldsymbol{\gamma}_{\lambda}\right) \\ &= -\frac{1}{2}\left[\frac{1}{\mathbf{g}}\left(-d\mathbf{g}\wedge\boldsymbol{\gamma}_{\lambda} + \mathbf{g}d\boldsymbol{\gamma}_{\lambda} + \mathbf{g}(\boldsymbol{\gamma}_{\alpha}\wedge\boldsymbol{\gamma}_{\lambda})\delta\boldsymbol{\gamma}^{\alpha} + \frac{1}{2}d\mathbf{g}\wedge\boldsymbol{\gamma}_{\lambda}\right)\right].\end{aligned} \tag{120}$$

Now, recalling again that the metric compatibility condition $D_{\kappa}g_{\lambda\rho} = 0$, we have

$$\begin{aligned}&\frac{1}{2\mathbf{g}}[d\mathbf{g}\wedge\boldsymbol{\gamma}_{\lambda} + 2\mathbf{g}\delta\boldsymbol{\gamma}^{\alpha}(\boldsymbol{\gamma}_{\alpha}\wedge\boldsymbol{\gamma}_{\lambda}] \\ &= \boldsymbol{\Gamma}_{\kappa}^{\kappa}\wedge\boldsymbol{\gamma}_{\lambda} + \delta\boldsymbol{\gamma}^{\alpha}(\boldsymbol{\gamma}_{\alpha}\wedge\boldsymbol{\gamma}_{\lambda}) \\ &= (\Gamma_{\beta\alpha\kappa} + \Gamma_{\alpha\beta\kappa})g^{\kappa\alpha}\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}_{\lambda} \\ &= (\partial_{\kappa}g_{\alpha\beta})g^{\kappa\alpha}\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}_{\lambda} \\ &= (dg_{\alpha\beta}\lrcorner\boldsymbol{\gamma}^{\beta})\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}_{\lambda},\end{aligned} \tag{121}$$

and Eq.(120) becomes

$$\mathcal{S}_{\lambda} = -\frac{1}{2}\left[\frac{1}{\mathbf{g}}\left(-d\mathbf{g}\wedge\boldsymbol{\gamma}_{\lambda} + \mathbf{g}d\boldsymbol{\gamma}_{\lambda}\right) + (dg_{\alpha\beta}\lrcorner\boldsymbol{\gamma}^{\beta})\boldsymbol{\gamma}^{\alpha}\wedge\boldsymbol{\gamma}_{\lambda}\right]. \tag{122}$$

However, we also have

$$-d\mathbf{g}\wedge\boldsymbol{\gamma}_\lambda+\mathbf{g}d\boldsymbol{\gamma}_\lambda = g_{\lambda\sigma}\mathbf{g}\left[-\partial_\beta(\ln\mathbf{g})g^{\nu\beta}g^{\sigma\rho}+g^{\beta\rho}\partial_\beta g^{\sigma\nu}\right]\boldsymbol{\gamma}_\nu\wedge\boldsymbol{\gamma}_\rho$$
$$=\frac{1}{2}g_{\lambda\sigma}\partial_\beta\left[\mathbf{g}\left(g^{\sigma\nu}g^{\rho\beta}-g^{\rho\sigma}g^{\nu\beta}\right)\right]\boldsymbol{\gamma}_\nu\wedge\boldsymbol{\gamma}_\rho$$
$$-\mathbf{g}(dg_{\alpha\beta}\lrcorner\boldsymbol{\gamma}^\beta)\boldsymbol{\gamma}^\alpha\wedge\boldsymbol{\gamma}_\lambda, \tag{123}$$

and finally we get

$$\mathcal{S}_\lambda=\frac{1}{2}\frac{1}{2(-\mathbf{g})}g_{\lambda\sigma}\partial_\beta\left[\mathbf{g}\left(g^{\sigma\nu}g^{\rho\beta}-g^{\rho\sigma}g^{\nu\beta}\right)\right]\boldsymbol{\gamma}_\nu\wedge\boldsymbol{\gamma}_\rho, \tag{124}$$

which gives an equivalent expression for the $\mathcal{S}_\lambda^{\nu\rho}$, which is very useful in calculations in GR, e.g., in the calculation of what is there defined as the "inertia" mass of a body creating a gravitational field. (see Eq.(54) and below)

$$\mathcal{S}_\lambda^{\nu\rho}=\frac{1}{2(-\mathbf{g})}g_{\lambda\sigma}\partial_\beta\left[\mathbf{g}\left(g^{\sigma\nu}g^{\rho\beta}-g^{\rho\sigma}g^{\nu\beta}\right)\right]. \tag{125}$$

From Eq.(115) above we can then write an equivalent formula for Freud's $\mathfrak{U}_\mu^{\lambda\sigma}$, namely:

$$\mathfrak{U}_\lambda^{\nu\rho}=\sqrt{-\mathbf{g}}\mathcal{S}_\lambda^{\nu\rho}=\frac{1}{2\sqrt{-\mathbf{g}}}g_{\lambda\sigma}\partial_\beta\left[\mathbf{g}\left(g^{\sigma\nu}g^{\rho\beta}-g^{\rho\sigma}g^{\nu\beta}\right)\right]$$
$$=-\frac{1}{2}\mathfrak{g}_{\lambda\sigma}\partial_\beta\left[\left(\mathfrak{g}^{\rho\sigma}\mathfrak{g}^{\nu\beta}-\mathfrak{g}^{\sigma\nu}\mathfrak{g}^{\rho\beta}\right)\right]. \tag{126}$$

C.3. The Freud Superpotentials $\mathbf{U}_\lambda$

We also introduce the Freud's superpotentials, i.e., the pseudo 2-forms $\mathbf{U}_\lambda\in\sec\bigwedge^2 T^*M$, by:

$$\mathbf{U}_\lambda=\frac{1}{2}\mathfrak{U}_\lambda^{\nu\rho}\in\boldsymbol{\gamma}_\nu\wedge\boldsymbol{\gamma}_\rho. \tag{127}$$

Now, Freud [8] defined in his Eq.(1)

$$\mathfrak{U}_\lambda^{\nu}=\partial_\rho\mathfrak{U}_\lambda^{\nu\rho}=-\sqrt{-\mathbf{g}}\Gamma_{\rho\kappa}^{\kappa}\mathcal{S}_\lambda^{\nu\rho}+\sqrt{-\mathbf{g}}\partial_\rho\mathcal{S}_\lambda^{\nu\rho}. \tag{128}$$

On the other hand from Eq.(32) we have

$$\star^{-1}d\star\mathcal{S}_\lambda=\delta\mathcal{S}_\lambda=-\boldsymbol{\partial}\lrcorner\mathcal{S}_\lambda=(-\partial_\nu\mathcal{S}_\lambda^{\nu\rho})\boldsymbol{\gamma}_\rho=-\mathcal{G}_\lambda-t_\lambda \tag{129}$$

or

$$-2\partial_\kappa\mathcal{S}_\nu^{\kappa\rho}=-2R_\nu^\rho+R\delta_\nu^\rho-2t_\nu^\rho. \tag{130}$$

Writing

$$\mathfrak{R}_\nu^\rho=\sqrt{-\mathbf{g}}R_\nu^\rho,\ \mathfrak{R}=\sqrt{-\mathbf{g}}R,\ \mathfrak{t}_\nu^\rho=\sqrt{-\mathbf{g}}t_\nu^\rho \tag{131}$$

and using Eq.(128) we have

$$-2\sqrt{-\mathbf{g}}\partial_\kappa\mathcal{S}_\nu^{\rho\kappa}+2\sqrt{-\mathbf{g}}\Gamma_{\alpha\kappa}^{\kappa}\mathcal{S}_\nu^{\rho\alpha}=2\mathfrak{R}_\nu^\rho-\mathfrak{R}\delta_\nu^\rho+2\sqrt{-\mathbf{g}}\Gamma_{\alpha\kappa}^{\kappa}\mathcal{S}_\nu^{\rho\alpha}+2\mathfrak{t}_\nu^\rho$$

or

$$\begin{aligned}2\mathfrak{U}_\nu^\rho &= -2\mathfrak{R}_\nu^\rho + \mathfrak{R}\delta_\nu^\rho - \frac{1}{2}\Gamma^\kappa_{\kappa\rho}\mathfrak{g}_{\lambda\sigma}\left[\left(\mathfrak{g}^{\sigma\nu}\mathfrak{g}^{\rho\beta} - \mathfrak{g}^{\rho\sigma}\mathfrak{g}^{\nu\beta}\right)\right]_{,\beta} - 2\mathfrak{t}_\nu^\rho \\ &= \delta_\nu^\rho(\mathfrak{R}+\mathfrak{L}) - 2\mathfrak{R}_\nu^\rho - \frac{1}{2}\Gamma^\kappa_{\kappa\rho}\mathfrak{g}_{\lambda\sigma}\left[\left(\mathfrak{g}^{\sigma\nu}\mathfrak{g}^{\rho\beta} - \mathfrak{g}^{\rho\sigma}\mathfrak{g}^{\nu\beta}\right)\right]_{,\beta} - 2\mathfrak{t}_\nu^\rho - \mathfrak{L}\delta_\nu^\rho, \end{aligned} \tag{132}$$

which can be written as [8]

$$2\mathfrak{U}_\lambda^\nu = \delta_\nu^\rho(\mathfrak{R}+\mathfrak{L}) - 2\mathfrak{R}_\nu^\rho + \left(\Gamma^\nu_{\mu\rho}\partial_\lambda\mathfrak{g}^{\mu\rho} - \Gamma^\kappa_{\kappa\mu}\partial_\lambda\mathfrak{g}^{\mu\nu}\right) \tag{133}$$

with

$$\mathfrak{L} = \mathfrak{g}^{\mu\nu}\left[\Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho}\right]. \tag{134}$$

Appendix D. The Einstein Energy-Momentum Pseudo 3-Forms $\star\mathfrak{e}^\lambda$

We have from Eq.(127)

$$\begin{aligned}\boldsymbol{\partial}\lrcorner\mathbf{U}_\lambda &= \boldsymbol{\partial}\lrcorner(\sqrt{-\mathbf{g}}\mathcal{S}_\lambda) = \boldsymbol{\gamma}^\kappa\lrcorner D_{\partial_\kappa}(\sqrt{-\mathbf{g}}\mathcal{S}_\lambda) \\ &= -\sqrt{-\mathbf{g}}\Gamma^\alpha_\alpha\lrcorner\mathcal{S}_\lambda + \sqrt{-\mathbf{g}}\boldsymbol{\partial}\lrcorner\mathcal{S}_\lambda \\ &= -\sqrt{-\mathbf{g}}\Gamma^\alpha_\alpha\lrcorner\mathcal{S}_\lambda + \sqrt{-\mathbf{g}}(\mathbf{T}_\lambda + t_\lambda), \end{aligned} \tag{135}$$

Defining $\mathfrak{T}_\lambda$ and $\mathfrak{t}_\lambda \in \sec\bigwedge^1 T^*M$ by

$$\mathfrak{T}_\lambda = \sqrt{-\mathbf{g}}\mathbf{T}_\lambda, \tag{136}$$

$$\mathfrak{t}_\lambda = \sqrt{-\mathbf{g}}(t_\lambda - \Gamma^\alpha_\alpha\lrcorner\mathcal{S}_\lambda) \tag{137}$$

or

$$\star\mathfrak{t}_\lambda = \sqrt{-\mathbf{g}}(\star t_\lambda + \Gamma^\kappa_\kappa \wedge \star\mathcal{S}_\lambda) \tag{138}$$

we get

$$\boldsymbol{\partial}\lrcorner\mathbf{U}_\lambda = \mathfrak{T}_\lambda + \mathfrak{t}_\lambda. \tag{139}$$

In components

$$\partial_\kappa\mathfrak{U}_\lambda^{\kappa\rho} = \mathfrak{T}_\lambda^\rho + \mathfrak{t}_\lambda^\rho \tag{140}$$

Comparing[28] Eq.(140) with Eq.(5-5.5) of [36] we see that

$$\mathfrak{t}_\lambda^\rho = \sqrt{-\mathbf{g}}(t_\lambda^\rho - \Gamma^\kappa_{\alpha\kappa}\mathcal{S}_\lambda^{\alpha\rho}) \tag{141}$$

is what is there called the components of the Einstein pseudo-tensor.

Comparing[29] Eq.(140) with Eq.(2.14) of [13] we see that what is there called the components of the Einstein pseudo-tensor are the $\mathfrak{e}_\lambda^\rho$ given by

$$\mathfrak{e}_\lambda^\rho = (t_\lambda^\rho - \Gamma^\kappa_{\alpha\kappa}\mathcal{S}_\lambda^{\alpha\rho}). \tag{142}$$

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[28]Take into account that our definition of the Ricci-tensor differs by a signal from the one of the quoted author.

[29]See previous footnote.

Also taking into account Eq.(30) we have for the Einstein 3-forms:

$$\star\mathfrak{e}^{\lambda} = \frac{1}{2}\mathbf{\Gamma}_{\alpha\beta} \wedge [\omega^{\lambda}_{\kappa} \wedge \star(\boldsymbol{\theta}^{\alpha} \wedge \boldsymbol{\theta}^{\beta} \wedge \boldsymbol{\theta}^{\kappa}) + \mathbf{\Gamma}^{\beta}_{\kappa} \wedge \star(\boldsymbol{\theta}^{\alpha} \wedge \boldsymbol{\theta}^{\kappa} \wedge \boldsymbol{\theta}^{\lambda}) + 2\mathbf{\Gamma}^{\kappa}_{\kappa} \lrcorner \mathcal{S}^{\lambda}]. \quad (143)$$

From this we see that Einstein superpotentials are nothing more than the Freud's superpotentials $\mathbf{U}_{\lambda}$.

Remark 9. The coordinate expression for $\mathfrak{e}^{\rho}_{\lambda}$ if you need it can be found in several books, e.g., [4, 13]. However, important from a historical point of view is to mention that already in 1917 the famous italian mathematician T. Levi-Civita[30] already pointed out [12] that Einstein solution for the energy-momentum description of the gravitational field (the pseudo tensor) was a nonsequitur.

D.1. Einstein "Inertial" Mass $m_{\mathbf{E}}$

In Section 4 the "inertial" mass of a body generating a gravitational field represented by a Lorentzian spacetime with metric $\mathbf{g}$ has been defined by $m_{\mathbf{I}} = \int \star S^{\mu}$, which we comment to be gauge dependent quantity. Using Eq.(139) we may define the Einstein "inertial mass" by

$$m_{\mathbf{E}} = \int_{S^2} \star \mathbf{U}^0. \quad (144)$$

where S^2 is a surface of radius $r \to \infty$. Let us calculate $\star\mathbf{U}_{\lambda}$ in a coordinate basis. Recalling Eq.(126) and Eq.(69) we have

$$\begin{aligned}
\star\mathbf{U}_{\lambda} &= -\frac{1}{2}\frac{1}{2\sqrt{-\mathbf{g}}} g_{\lambda\sigma}\partial_{\beta}\left[\mathbf{g}\left(g^{\rho\sigma}g^{\nu\beta} - g^{\sigma\nu}g^{\rho\beta}\right)\right] \star (\boldsymbol{\gamma}_{\nu} \wedge \boldsymbol{\gamma}_{\rho}) \\
&= -\frac{1}{2}\frac{1}{2\sqrt{-\mathbf{g}}} g_{\lambda\sigma}g_{\nu\mu}g_{\rho\varkappa}\partial_{\beta}\left[\mathbf{g}\left(g^{\rho\sigma}g^{\nu\beta} - g^{\sigma\nu}g^{\rho\beta}\right)\right] \star (\boldsymbol{\gamma}^{\mu} \wedge \boldsymbol{\gamma}^{\varkappa}) \\
&= -\frac{1}{2}\frac{1}{2}\frac{\sqrt{-\mathbf{g}}}{2\sqrt{-\mathbf{g}}} g_{\lambda\sigma}g_{\nu\mu}g_{\rho\varkappa}\partial_{\beta}\left[\mathbf{g}\left(g^{\rho\sigma}g^{\nu\beta} - g^{\sigma\nu}g^{\rho\beta}\right)\right] g^{\mu\varepsilon}g^{\varkappa\tau}\epsilon_{\varepsilon\tau\alpha\omega}\boldsymbol{\gamma}^{\alpha} \wedge \boldsymbol{\gamma}^{\omega} \\
&= -\frac{1}{8} g_{\lambda\sigma}\partial_{\beta}\left[\mathbf{g}\left(g^{\rho\sigma}g^{\nu\beta} - g^{\sigma\nu}g^{\rho\beta}\right)\right]\epsilon_{\nu\rho\alpha\omega}\boldsymbol{\gamma}^{\alpha} \wedge \boldsymbol{\gamma}^{\omega}. \quad (145)
\end{aligned}$$

Now, for a diagonal metric tensor we have (with $k, m, n = 1, 2, 3$)

$$\begin{aligned}
\star\mathbf{U}_0 &= \frac{1}{4} g_{00}\partial_{\beta}\left[\mathbf{g}\left(g^{00}g^{\rho\beta}\right)\right]\epsilon_{\rho0\alpha\omega}\boldsymbol{\gamma}^{\alpha} \wedge \boldsymbol{\gamma}^{\omega} \quad (146) \\
&= -\frac{1}{4} g_{00}\partial_{l}(-\mathbf{g}g^{00}g^{kl})\epsilon_{0kmn}\boldsymbol{\gamma}^{m} \wedge \boldsymbol{\gamma}^{n}.
\end{aligned}$$

$$\star\mathbf{U}^0 = g^{00} \star \mathbf{U}_0 = -\frac{1}{4}\partial_{l}(-g_{11}g_{22}g_{33}g^{kl})\epsilon_{0kmn}\boldsymbol{\gamma}^{m} \wedge \boldsymbol{\gamma}^{n}. \quad (147)$$

Taking into account that if we use "Cartesian like coordinates" $\{x^{\mu}\}$ (as, e.g., in the *isotropic* form[31] of the Schwarzschild solution [38]) we must define the radial variable of the standard spherical coordinates (r, θ, φ) by. $r^2 = -g_{ij}x^i x^j$.

[30] Yes, the one that gives name to the connection used in GR.

[31] In isotropic Cartesian coordinates the Schwarzschild solution of the Einstein-Hilbert equation reads (with $r_g = 2mG/c^2$ in MKS units): $\mathbf{g} = \left(\frac{1-r_g/4r}{1+r_g/4r}\right)^2 dt \otimes dt - (1 + r_g/4r)^2 \sum_{i=1}^{3} dx^i \otimes dx^i$.

We parametrize (as it is standard) the surface S^2 which has equation $f = x^i x_i + r^2 = 0$ with the coordinates (θ, φ). The "Euclidean" unitary vector normal to this surface has thus the components (n_1, n_2, n_3) with $n_k = -\frac{x_k}{r}$. Now, we have

$$\begin{aligned}\star\mathbf{U}^0 &= -\frac{1}{4}\partial_l(-g_{11}g_{22}g_{33}g^{kl})\epsilon_{0kmn}dx^m \wedge dx^n \\ &= -\frac{1}{2}(U^1 dx^2 \wedge dx^3 + U^2 dx^3 \wedge dx^1 + U^3 dx^1 \wedge dx^2),\end{aligned} \tag{148}$$

with

$$U^k = \partial_l(-g_{11}g_{22}g_{33}g^{kl}). \tag{149}$$

Since

$$dx^i = \frac{\partial x^i}{\partial \theta}d\theta + \frac{\partial x^i}{\partial \varphi}d\varphi \tag{150}$$

we can write Eq.(148) as

$$\begin{aligned}\star\mathbf{U}^0 &= -\frac{1}{2}\det\begin{bmatrix} U^1 & U^2 & U^3 \\ \frac{\partial x^1}{\partial \theta} & \frac{\partial x^2}{\partial \theta} & \frac{\partial x^3}{\partial \theta} \\ \frac{\partial x^1}{\partial \varphi} & \frac{\partial x^2}{\partial \varphi} & \frac{\partial x^3}{\partial \varphi}\end{bmatrix} d\theta \wedge d\varphi \\ &= -\frac{1}{2}r^2 \sin^2\theta \det\begin{bmatrix} U^1 & U^2 & U^3 \\ \frac{\partial x^1}{r\partial \theta} & \frac{\partial x^2}{r\partial \theta} & \frac{\partial x^3}{r\partial \theta} \\ \frac{\partial x^1}{r\sin^2\theta\partial \varphi} & \frac{\partial x^2}{r\sin^2\theta\partial \varphi} & \frac{\partial x^3}{r\sin^2\theta\partial \varphi}\end{bmatrix} d\theta \wedge d\varphi\end{aligned} \tag{151}$$

Then putting $\vec{U} = (U^1, U^2, U^3)$ and defining moreover the euclidean orthonormal vectors

$$\begin{aligned}\vec{e}_r &= (n_1, n_2, n_3) \\ \vec{e}_\theta &= (\frac{1}{r}\frac{\partial x^1}{\partial \theta}, \frac{1}{r}\frac{\partial x^2}{\partial \theta}, \frac{1}{r}\frac{\partial x^3}{\partial \theta}), \\ \vec{e}_\varphi &= (\frac{1}{r\sin^2\theta}\frac{\partial x^1}{\partial \varphi}, \frac{1}{r\sin^2\theta}\frac{\partial x^2}{\partial \varphi}, \frac{1}{r\sin^2\theta}\frac{\partial x^3}{\partial \varphi}),\end{aligned} \tag{152}$$

we can rewrite Eq.(151) using the standard notation of vector calculus[32] as:

$$\begin{aligned}\star\mathbf{U}^0 &= -\frac{1}{2}r^2\sin^2\theta\ \vec{U} \bullet (\vec{e}_\theta \times \vec{e}_\varphi)d\theta \wedge d\varphi \\ &= -\frac{1}{2}r^2\sin^2\theta\ (\vec{U} \bullet \vec{e}_r)d\theta \wedge d\varphi \\ &= -\frac{1}{2}r^2\sin^2\theta U^i n_i d\theta \wedge d\varphi \\ &= -\frac{1}{2}\partial_l(-g_{11}g_{22}g_{33}g^{li})n_i r^2 \sin^2\theta d\theta \wedge d\varphi.\end{aligned} \tag{153}$$

[32]With $\bullet$ denoting the euclidean scalar product and $\times$ the vector product.

Finally, making the radius $r \to \infty$ we get

$$\begin{aligned} m_{\mathbf{E}} &= \int_{S^2} \star \mathbf{U}^0 = - \lim_{r\to\infty} \frac{1}{2} \int_{S^2} \partial_l(-g_{11}g_{22}g_{33}g^{lk}) n_k r^2 \sin^2\theta d\theta \wedge d\varphi \\ &= -\frac{1}{2} \lim_{r\to\infty} \int_{S^2} \frac{\partial}{\partial x^l}(-g_{11}g_{22}g_{33}g^{kl}) n_k r^2 \sin^2\theta d\theta d\varphi \\ &= \frac{1}{2} \lim_{r\to\infty} \int_{S^2} \frac{x_k}{r} \frac{\partial}{\partial x^l}(-g_{11}g_{22}g_{33}g^{kl}) r^2 \sin^2\theta d\theta d\varphi, \end{aligned} \tag{154}$$

a well known result.

For the *isotropic* form of the Schwarzschild metric a simple calculation shows that $m_{\mathbf{E}} = m$, the parameter identified as "gravitational" mass in the solution of Einstein's equations.

Appendix E. Landau-Lifshitz Energy-Momentum Pseudo 3-Forms $\star\mathfrak{l}_\lambda$

Given a coordinate basis associated with a chart with coordinates $\{x^\alpha\}$ covering $U \subset M$ and writing $t^\mu = t^{\mu\nu}\gamma_\nu$ given by Eq.(35), we immediately discover that the $t^{\mu\nu}$ are not symmetric. So, this object, cannot be used to formulate a *chart dependent*[33] angular momentum "conservation law" for system consisted of matter plus the gravitational field, i.e., the objects $M^{\mu\nu} \in \sec \bigwedge^3 T^*M$ given by

$$M^{\mu\nu} = x^\mu(\star\mathbf{T}^\mu + \star t^\mu) - x^\nu(\star\mathbf{T}^\nu + \star t^\nu). \tag{155}$$

In view of this fact let us find an energy-momentum "conservation law" involving a *symmetric* energy-momentum pseudo tensor.

Define the superpotentials

$$\mathbf{H}^\mu = \mathbf{g}\mathcal{S}^\mu = -\sqrt{-\mathbf{g}}\mathbf{U}^\mu. \tag{156}$$

Then we have

$$\begin{aligned} \boldsymbol{\partial}\lrcorner(\mathbf{H}^\mu) &= \mathbf{g}\boldsymbol{\partial}\lrcorner\mathcal{S}^\mu + 2\mathbf{g}\boldsymbol{\Gamma}^\kappa_\kappa\lrcorner S^\mu \\ &= (-\mathbf{g})(\mathbf{T}^\mu - t^\mu - 2\boldsymbol{\Gamma}^\kappa_\kappa\lrcorner S^\mu) \qquad (157) \\ &= (-\mathbf{g})(\mathbf{T}^\mu - \mathfrak{l}^\mu), \qquad (158) \end{aligned}$$

where

$$\begin{aligned} \star\mathfrak{l}^\mu &= (\star t^\mu + 2\boldsymbol{\Gamma}^\kappa_\kappa\lrcorner S^\mu) \\ &= (\star t^\mu - 2\boldsymbol{\Gamma}^\kappa_\kappa \wedge \star S^\mu), \end{aligned} \tag{159}$$

are the Landau-Lifshitz energy-momentum 3-forms as it is obvious comparing Eq.(73) with Eq.(96.15) of [11]. Also, taking into account Eq.(35) we have

$$\star\mathfrak{l}^\mu = \frac{1}{2}\boldsymbol{\Gamma}_{\alpha\beta} \wedge [\omega^\mu_\kappa \wedge \star(\boldsymbol{\theta}^\alpha \wedge \boldsymbol{\theta}^\beta \wedge \boldsymbol{\theta}^\kappa) + \boldsymbol{\Gamma}^\beta_\kappa \wedge \star(\boldsymbol{\theta}^\alpha \wedge \boldsymbol{\theta}^\kappa \wedge \boldsymbol{\theta}^\mu) + \boldsymbol{\Gamma}^\kappa_\kappa\lrcorner\mathcal{S}^\mu]. \tag{160}$$

[33] It is possible to define global angular momentum 3-forms only for particular Lorentzian spacetimes.

However, the components $\mathfrak{t}^{\mu\nu}$ are symmetric [11], as may be verified by a long calculation.

E.1. Landau-Lifshitz "Inertial" Mass $m_{\mathbf{LL}}$

As a last observation, taking into account Eq.(154) if we compute

$$m_{\mathbf{LL}} = \int_{S^2} \star \mathbf{H}^{\mu}$$

on the surface of a sphere of radius r and making the radius $r \to \infty$ we get for the Schwarzschild solution (in Cartesian isotropic coordinates) and taking into account that $\lim_{r\to\infty} \sqrt{-\mathbf{g}} = 1$,

$$m_{\mathbf{LL}} = \frac{1}{2} \lim_{r\to\infty} \int_{S^2} \frac{x_k}{r} \frac{\partial}{\partial x^l} (-g_{11} g_{22} g_{33} g^{kl}) r^2 \sin^2 \theta d\theta d\varphi = m_{\mathbf{E}} = m \tag{161}$$

At this point we end this long Appendix with a comment by Logunov [13]:

"it was the fact that "inertial" mass coincides with gravitational mass that gave grounds for asserting that they are equal in GR, to".

Indeed, in their celebrated textbook, Landau and Lifshitz [11] wrote at page 334:

"... $P^0 = m$, a result[34] which was naturally to be expected. It is an expression of the equality of "gravitational" and "inertial"mass ("gravitational" mass is the mass that determine the gravitational field produced by the body, the same mass that appears in the metric tensor of the gravitational field, or in particular, in Newton's law; "inertial" mass is the mass that determines the ratio of energy momentum of the body; in particular, the rest energy of the body is equal to the mass multiplied by c^2."

However, as discussed in [2, 13] the $\int_{S^2} \star \mathbf{H}^0$ (or $\int_{S^2} \star \mathbf{U}^0$) being the integral of a gauge dependent quantity depends on the coordinate chart chosen for its computation, and we can easily build examples in which the "inertial" mass is *different* from the "gravitational" mass, violating the main Einstein's heuristic guide to GR, namely the equality of both masses. This results makes one to understand the reason of Sachs & Wu statement quoted above.

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