On the 2nd Bianchi identity

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Though the Cartan version of the 2nd Bianchi identity

(1) $$\mathbf{D \wedge R^{a}{}_{b} := d \wedge R^{a}{}_{b} + \omega^{a}{}_{c} \wedge R^{c}{}_{b} - \omega^{c}{}_{b} \wedge R^{a}{}_{c} = 0\,.}$$

is well-known in literature [1, p.93 (3.141)], [2, p.489 (J.32)], [3, p.208 (C.1.69)], [4, p.123 (4,127 b)] there are also publications [5, p.13] where this result is doubted with the remark that "the torsion is missing incorrectly". The author of [5] believes that (1) is valid only for the *symmetric* Levi-Civita connection. Therefore we repeat here the simple algebra proof of eq. (1) for general torsion from literature:

Proof of the 2nd Bianchi identity (1)

The curvature form is defined by

(2) $$R^{a}{}_{b} = d \wedge \omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}\,.$$

Due to the Poincaré Lemma on exact differential forms we have

(3) $$d \wedge (d \wedge \omega^{a}{}_{b}) = 0$$

to obtain by applying the Leibniz rule:

(4) $$d\wedge R^{a}{}_{b} = d\wedge(d\wedge\omega^{a}{}_{b} + \omega^{a}{}_{c}\wedge\omega^{c}{}_{b}) = 0 + d\wedge(\omega^{a}{}_{c}\wedge\omega^{c}{}_{b}) = \omega^{c}{}_{b}\wedge d\wedge\omega^{a}{}_{c} - \omega^{a}{}_{c}\wedge d\wedge\omega^{c}{}_{b}$$

On the other hand we may conclude:

(5) $$\left.\begin{array}{l} \omega^{a}{}_{c}\wedge R^{c}{}_{b} = \omega^{a}{}_{c}\wedge(d\wedge\omega^{c}{}_{b} + \omega^{c}d\wedge\omega^{d}{}_{b}) \\ \omega^{c}{}_{b}\wedge R^{a}{}_{c} = \omega^{c}{}_{b}\wedge(d\wedge\omega^{a}{}_{c} + \omega^{a}{}_{d}\wedge\omega^{d}{}_{c}) \end{array}\right\} \Rightarrow \quad \omega^{c}{}_{b}\wedge d\wedge\omega^{a}{}_{c} - \omega^{a}{}_{c}\wedge d\wedge\omega^{c}{}_{b} = \omega^{c}{}_{b}\wedge R^{a}{}_{c} - \omega^{a}{}_{c}\wedge R^{c}{}_{b}\,,$$

hence

(6) $$d \wedge R^{a}{}_{b} = \omega^{c}{}_{b} \wedge R^{a}{}_{c} - \omega^{a}{}_{c} \wedge R^{c}{}_{b}\,,$$

or by introducing the exterior derivative $D \wedge R^{a}{}_{b}$

(7) $$\mathbf{D \wedge R^{a}{}_{b} := d \wedge R^{a}{}_{b} + \omega^{a}{}_{c} \wedge R^{c}{}_{b} - \omega^{c}{}_{b} \wedge R^{a}{}_{c} = 0\,.}$$

References

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[3] F.W. Hehl and Y.N. Obukhov, *Foundations of Classical Electrodynamics - Charge, Flux, and Metric*, Birkhäuser 2003, ISBN 0-8176-4222-6

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