Riemann-Cartan Connection and its Decomposition. One More Assessment of "ECE Theory"

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Abstract

In this short pedagogical note we clarify some subtleties concerning the symmetries of the *coefficients* of a Riemann-Cartan *connection* and the symmetries of the coefficients of the *contorsion* tensor that has been a source of some confusion in the literature, in particular in a so called 'ECE theory'. We show in details that the coefficients of the contorsion tensor of a Riemann-Cartan connection has a symmetric part and an antisymmetric part, the symmetric part defining the *strain tensor* of the connection. Moreover, the contorsion tensor has also a *bastard* anti-symmetry when written with all its indices in the 'covariant' positions.

1 Some Preliminaries

Let M be a 4-dimensional Hausdorff, paracompact and locally compact manifold admitting a Lorentzian metric tensor $\boldsymbol{g} \in \sec T_0^2 M$. Let us suppose that M is also spacetime orientable by a global 4-form field $\tau_{\boldsymbol{g}} \in \sec \bigwedge^4 T^*M$ and also time orientable[1] by the relation $\uparrow$ and let be $\mathring{D}$ the Levi-Civita connection of $\boldsymbol{g}$. Under these conditions we call the pentuple $\langle M, \boldsymbol{g}, \mathring{D}, \tau_{\boldsymbol{g}}, \uparrow\rangle$ a *Lorentzian spacetime*. The curvature tensor of $\mathring{D}$ will be denoted in what follows by $\mathring{\boldsymbol{R}}$.

Let D be a general Riemann-Cartan connection on M, *i.e.*, $D\boldsymbol{g} = 0$. In general the Riemann (curvature) tensor $\boldsymbol{R}$ and the torsion tensor $\boldsymbol{\Theta}$ of D are non

[1] See, e.g., [12] for details.

null. Under the conditions of orientability and time orientability the pentuple $\langle M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow\rangle$is said to be a Riemann-Cartan spacetime.

By definition a manifold equipped with a given connection is said to be *flat* if the Riemann (curvature) tensor of that connection is null.

Let $\langle x^\mu\rangle$ and $\langle x'^\mu\rangle$ be respectively coordinate functions for $U \subset M$ and $U' \subset M$ such that $U \cap U' \neq \varnothing$.

Moreover, let $\langle \boldsymbol{e}_\mu = \partial/\partial x^\mu\rangle$ and $\langle \boldsymbol{e}'_\mu = \partial/\partial x'^\mu\rangle$ be respectively basis of TU and TU' ($\mu = 0, 1, 2, 3$) and $\langle \vartheta^\mu = dx^\mu\rangle$ and $\langle \vartheta'^\mu = dx'^\mu\rangle$ the corresponding dual basis i.e., basis for T^*U and T^*U'. We also introduce the reciprocal basis $\langle \boldsymbol{e}^\mu\rangle$ of $\langle \boldsymbol{e}_\mu\rangle$ and $\langle \boldsymbol{e}'^\mu\rangle$ of $\langle \boldsymbol{e}'_\mu\rangle$ for TU and TU' and the reciprocal basis $\langle \vartheta_\mu\rangle$ of $\langle \vartheta^\mu\rangle$ and $\langle \vartheta'_\mu\rangle$ of $\langle \vartheta'^\mu\rangle$ for T^*U and T^*U', such that

$$\begin{aligned} \boldsymbol{g} &= g_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu = g^{\mu\nu}\vartheta_\mu \otimes \vartheta_\nu, \quad g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu, \\ \boldsymbol{e}^\mu &= g^{\mu\nu}\boldsymbol{e}_\nu, \quad \vartheta_\mu = g_{\mu\nu}\vartheta^\nu, \text{etc.} \end{aligned} \tag{1}$$

Moreover we introduce as metric for the cotangent bundle the object $g \in \sec T^0_2 M$,

$$g = g^{\mu\nu}\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu = g_{\mu\nu}\boldsymbol{e}^\mu \otimes \boldsymbol{e}^\nu$$

and define the scalar product of arbitrary of arbitrary vector vector fields $\mathbf{V}, \mathbf{W} \in \sec TM$ and arbitrary 1-form fields $\boldsymbol{X}$, $\boldsymbol{Y} \in \sec \bigwedge^1 T^*M$ by

$$\mathbf{V} \cdot \mathbf{W} = \boldsymbol{g}(\mathbf{V}, \mathbf{W}), \quad \boldsymbol{X} \cdot \boldsymbol{Y} = g(\boldsymbol{X}, \boldsymbol{Y}). \tag{2}$$

We write

$$\begin{aligned} D_{\boldsymbol{e}_\mu}\vartheta^\nu &:= -\Gamma^{\cdot\cdot\nu}_{\mu\alpha\cdot}\vartheta^\alpha, \quad D_{\boldsymbol{e}'_\mu}\vartheta'^\nu := -\Gamma'^{\cdot\cdot\nu}_{\mu\alpha\cdot}\vartheta'^\alpha, \\ \mathring{D}_{\boldsymbol{e}_\mu}\vartheta^\nu &:= -\mathring{\Gamma}^{\cdot\cdot\nu}_{\mu\alpha\cdot}\vartheta^\alpha, \quad \mathring{D}_{\boldsymbol{e}'_\mu}\vartheta'^\nu := -\mathring{\Gamma}'^{\cdot\cdot\nu}_{\mu\alpha\cdot}\vartheta'^\alpha. \end{aligned} \tag{3}$$

2 $\Gamma^{\cdot\cdot\mu}_{\rho\sigma\cdot}$ is not in General Antisymmetric in the Lower Indices

As it is well known, given an arbitrary connection D, the relation between $\Gamma'^{\cdot\cdot\lambda}_{\iota\kappa\cdot}$ and $\Gamma^{\cdot\cdot\mu}_{\rho\sigma\cdot}$ (dubbed transformation law for the connection coefficients) is

$$\Gamma'^{\cdot\cdot\lambda}_{\iota\kappa\cdot} = \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial x^\rho}{\partial x'^\iota}\frac{\partial x^\sigma}{\partial x'^\kappa}\Gamma^{\cdot\cdot\mu}_{\rho\sigma\cdot} + \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x'^\iota \partial x'^\kappa} \tag{4}$$

From Eq.(4) we see that even if happens that $\Gamma^{\cdot\cdot\mu}_{\rho\sigma\cdot}$ is *antisymmetric* in a given coordinate basis, i.e., $\Gamma^{\cdot\cdot\mu}_{\rho\sigma\cdot} = -\,\Gamma^{\cdot\cdot\mu}_{\sigma\rho\cdot}$, in general it will be ***not*** antisymmetric in another coordinate chart since the term $\frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x'^\iota \partial x'^\kappa}$ is symmetric in the lower indices. This immediately contradicts the main claim of the so called "ECE unified field theory" [2], where it is stated that the connections coefficients of a Riemann-Cartan connection are *always* antisymmetric.

3 Relation between $\Gamma_{\mu\nu\cdot}^{\cdot\cdot\lambda}$ and $\mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\lambda}$

We shall prove that :

$$\Gamma_{\mu\nu\cdot}^{\cdot\cdot\lambda} = \mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\lambda} + K_{\mu\nu\cdot}^{\cdot\cdot\lambda} \tag{5}$$

where[3]

$$\begin{aligned} K_{\mu\nu\cdot}^{\cdot\cdot\beta} &:= \frac{1}{2}(T_{\mu\nu\cdot}^{\cdot\cdot\beta} + S_{\mu\nu\cdot}^{\cdot\cdot\beta}) \\ &= \frac{1}{2}g^{\lambda\beta}g_{\lambda\alpha}T_{\mu\nu\cdot}^{\cdot\cdot\alpha} - \frac{1}{2}g^{\beta\lambda}g_{\nu\rho}T_{\mu\sigma\cdot}^{\cdot\cdot\rho} - \frac{1}{2}g^{\beta\lambda}g_{\mu\alpha}T_{\nu\lambda\cdot}^{\cdot\cdot\alpha} \\ &= \frac{1}{2}(T_{\mu\nu\cdot}^{\cdot\cdot\beta} - T_{\nu\cdot\mu}^{\cdot\beta\cdot} + T_{\cdot\mu\nu}^{\beta\cdot\cdot}). \end{aligned} \tag{6}$$

and

$$T_{\mu\nu\cdot}^{\cdot\cdot\lambda} = \Gamma_{\mu\nu\cdot}^{\cdot\cdot\lambda} - \Gamma_{\nu\mu\cdot}^{\cdot\cdot\lambda} = -T_{\nu\mu\cdot}^{\cdot\cdot\lambda}, \tag{7}$$

$$S_{\mu\nu\cdot}^{\cdot\cdot\lambda} = -g^{\lambda\sigma}(g_{\nu\alpha}T_{\mu\sigma}^{\alpha} + g_{\mu\alpha}T_{\nu\sigma}^{\alpha}) = S_{\nu\mu\cdot}^{\cdot\cdot\lambda} \ . \tag{8}$$

Before presenting the proof (see also [14, 10]) of the above equations we recall that the $T_{\mu\nu\cdot}^{\cdot\cdot\lambda}$ are the components of the so called *torsion tensor*

$$\mathbf{\Theta} = \frac{1}{2}T_{\mu\nu\cdot}^{\cdot\cdot\lambda}\vartheta^{\mu} \wedge \vartheta^{\nu} \otimes \boldsymbol{e}_{\lambda} \in \sec \bigwedge{}^{2}T^{*}M \otimes TM, \tag{9}$$

Also, the $K_{\mu\nu\cdot}^{\cdot\cdot\lambda}$ are the components of an object that (since Schouten [13]) is called the con*torsion tensor*

$$\boldsymbol{K} = K_{\mu\nu\cdot}^{\cdot\cdot\lambda}\vartheta^{\mu} \otimes \vartheta^{\nu} \otimes \boldsymbol{e}_{\lambda} = K_{\mu\nu\lambda}^{\cdot\cdot\cdot}\vartheta^{\mu} \otimes \vartheta^{\nu} \otimes \boldsymbol{e}^{\lambda} \in \sec T^{*}M \otimes T^{*}M \otimes TM. \tag{10}$$

As can be easily verified from Eq.(6) it is the case that

$$K_{\mu\nu\lambda}^{\cdot\cdot\cdot} = g_{\lambda\alpha}K_{\mu\nu\cdot}^{\cdot\cdot\alpha} = -K_{\mu\lambda\nu}^{\cdot\cdot\cdot} \tag{11}$$

The validity of Eq.(11) lead many authors to say the contortion tensor is antisymmetric in the two last indices. However, it is necessary to observe here that (parodying Göckeler and Schücker [5]) the anti-symmetry is a *bastard* one,

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[2] See the criticisms to ECE theory in the list of references. Particularly, see [1] where Bruhn, pedagogicaly identifies that the mistake of the author of ECE papers regarding his statement that for any general connection its coefficients in any coordinate basis must be anti-symmetric is simply due to the fact that he did not know (until today) that a general real $n \times n$ matrix can be decomposed in a symmetric matrix plus an anti-symmetric one.

[3] Note that this formula differs by a factor of 1/2 and signal from the one in [6].

since we are comparing the components of $\boldsymbol{K}$ that live on different spaces, namely T^*M and TM.

Moreover, writing

$$\boldsymbol{K} = K_{\mu\cdot\nu}^{\cdot\beta\cdot}\vartheta^\mu \otimes \vartheta_\beta \otimes \boldsymbol{e}^\nu \in \sec T^*M \otimes TM \otimes T^*M. \tag{12}$$

where like in [6]

$$K_{\mu\cdot\nu}^{\cdot\beta\cdot} := g_{\nu\lambda} g^{\beta\kappa} K_{\mu\kappa\cdot}^{\cdot\cdot\lambda},$$

we have again a *bastard* anti-symmetry since

$$K_{\mu\cdot\nu}^{\cdot\beta\cdot} = -K_{\mu\nu\cdot}^{\cdot\cdot\beta}. \tag{13}$$

Finally we remark that (since Schouten [9]) the $S_{\mu\nu\cdot}^{\cdot\cdot\lambda} = S_{\nu\mu\cdot}^{\cdot\cdot\lambda}$ are said to be the components of the *strain tensor* (of the connection D)

$$\boldsymbol{S} = S_{\mu\nu\cdot}^{\cdot\cdot\lambda}\vartheta^\mu \otimes \vartheta^\nu \otimes \boldsymbol{e}_\lambda = S_{\mu\nu\lambda}^{\cdot\cdot\cdot}\vartheta^\mu \otimes \vartheta^\nu \otimes \boldsymbol{e}^\lambda \in \sec T^*M \otimes T^*M \otimes TM. \tag{14}$$

Remark *It is obvious from the above formulas that the contorsion tensor is not antisymmetric in the lower indices* $\mu\nu$ *due to the presence of the strain tensor that is symmetric contrary to what is stated, e.g., in* [1].

4 Proof of Eq.(5)

We start remembering that since $\mathring{D}\boldsymbol{g} = 0$ and $D\boldsymbol{g} = 0$ we can write in an arbitrary coordinate basis that:

$$\mathring{D}_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\rho} g_{\rho\lambda} - \mathring{\Gamma}_{\mu\lambda\cdot}^{\cdot\cdot\rho} g_{\nu\rho} = 0, \tag{15}$$

$$D_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho} g_{\rho\lambda} - \Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} g_{\nu\rho} = 0. \tag{16}$$

From Eq.(15) and some trivial algebra we get, as well known

$$\mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\rho} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right). \tag{17}$$

From Eq.(16) we can write:

$$\partial_\mu g_{\nu\lambda} = \Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho} g_{\rho\lambda} + \Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} g_{\nu\rho}, \tag{18}$$

$$\partial_\nu g_{\mu\lambda} = \Gamma_{\nu\mu\cdot}^{\cdot\cdot\rho} g_{\rho\lambda} + \Gamma_{\nu\lambda\cdot}^{\cdot\cdot\rho} g_{\mu\rho},, \tag{19}$$

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho} g_{\rho\nu} + \Gamma_{\lambda\nu\cdot}^{\cdot\cdot\rho} g_{\mu\rho}. \tag{20}$$

Then,

$$\begin{aligned} &\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \\ &= g_{\rho\lambda}(\Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho} + \Gamma_{\nu\mu\cdot}^{\cdot\cdot\rho}) + g_{\nu\rho}(\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} - \Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}) + g_{\mu\rho}(\Gamma_{\nu\lambda\cdot}^{\cdot\cdot\rho} - \Gamma_{\lambda\nu\cdot}^{\cdot\cdot\rho}). \end{aligned} \tag{21}$$

Observe that $\Gamma_{(\mu\nu)\cdot}^{\cdot\cdot\rho} : +\frac{1}{2}(\Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho} + \Gamma_{\nu\mu\cdot}^{\cdot\cdot\rho})$ is the symmetric part of $\Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho}$ whereas $\frac{1}{2}(\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} - \Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}) = \frac{1}{2}T_{\mu\lambda\cdot}^{\cdot\cdot\rho}$ is the antisymmetric part of $\Gamma_{\mu\rho\cdot}^{\cdot\cdot\rho}$. We can rearrange the terms in Eq.(21) taking into account the definition of the connections coefficients of the Levi-Civita connection $\mathring{D}$ as:

$$\begin{aligned} g_{\rho\lambda}\Gamma_{(\mu\nu)}^{\cdot\cdot\rho} &= \frac{1}{2}g_{\rho\lambda}(\Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho} + \Gamma_{\nu\mu\cdot}^{\cdot\cdot\rho}) \\ &= \frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) - \frac{1}{2}g_{\nu\rho}(\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} - \Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}) - g_{\rho\lambda}\frac{1}{2}(\Gamma_{\nu\lambda\cdot}^{\cdot\cdot\rho} - \Gamma_{\lambda\nu\cdot}^{\cdot\cdot\rho}). \end{aligned} \tag{22}$$

Then,

$$\begin{aligned} \Gamma_{(\mu\nu)}^{\cdot\cdot\beta} &= \frac{1}{2}g^{\beta\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\ &\quad - g^{\beta\lambda}\frac{1}{2}g_{\nu\rho}(\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} - \Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}) - g^{\beta\lambda}g_{\rho\lambda}\frac{1}{2}(\Gamma_{\nu\lambda\cdot}^{\cdot\cdot\rho} - \Gamma_{\lambda\nu\cdot}^{\cdot\cdot\rho}) \\ &= \mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\beta} - \frac{1}{2}g^{\beta\lambda}(g_{\nu\rho}T_{\mu\lambda\cdot}^{\cdot\cdot\rho} + g^{\beta\lambda}g_{\rho\mu}T_{\nu\lambda\cdot}^{\cdot\cdot\rho}) \\ &= \mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\beta} + \frac{1}{2}S_{\mu\lambda\cdot}^{\cdot\cdot\beta} \end{aligned} \tag{23}$$

Finally, taking into account that $\Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho} = \Gamma_{(\mu\nu)\cdot}^{\cdot\cdot\rho} + \Gamma_{[\mu\nu]\cdot}^{\cdot\cdot\rho} = \Gamma_{(\mu\nu)\cdot}^{\cdot\cdot\rho} + \frac{1}{2}T_{\mu\nu\cdot}^{\cdot\cdot\rho}$ we have using Eq.(23)

$$\Gamma_{\mu\nu\cdot}^{\cdot\cdot\rho} = \mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\rho} + \frac{1}{2}S_{\mu\lambda\cdot}^{\cdot\cdot\rho} + \frac{1}{2}T_{\mu\nu\cdot}^{\cdot\cdot\rho} = \mathring{\Gamma}_{\mu\nu\cdot}^{\cdot\cdot\rho} + K_{\mu\nu\cdot}^{\cdot\cdot\rho}, \tag{24}$$

and Eq.(5) is proved■

5 Relation Between the Curvature Tensors $\boldsymbol{R}$ and $\mathring{\boldsymbol{R}}$

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \sec TU$ and $\boldsymbol{\alpha} \in \sec \bigwedge^1 T^*U$. The curvature operators of $\mathring{D}$ and D are defined by

$$\mathring{\boldsymbol{\rho}}, \boldsymbol{\rho} : \sec TM \otimes TM \otimes TM \longrightarrow \sec TM, \tag{25}$$

$$\mathring{\boldsymbol{\rho}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \mathring{D}_{\boldsymbol{u}}\mathring{D}_{\boldsymbol{v}}\boldsymbol{w} - \mathring{D}_{\boldsymbol{v}}\mathring{D}_{\boldsymbol{u}}\boldsymbol{w} - \mathring{D}_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}, \tag{26}$$

$$\boldsymbol{\rho}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = D_{\boldsymbol{u}}D_{\boldsymbol{v}}\boldsymbol{w} - D_{\boldsymbol{v}}D_{\boldsymbol{u}}\boldsymbol{w} - D_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}. \tag{27}$$

It is usual to write [3] $\mathring{\boldsymbol{\rho}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \mathring{\boldsymbol{\rho}}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w}$, $\boldsymbol{\rho}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\rho}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w}$ and even to call curvature operators the objects

$$\begin{aligned} \mathring{\boldsymbol{\rho}}(\boldsymbol{u}, \boldsymbol{v}) &= \mathring{D}_{\boldsymbol{u}}\mathring{D}_{\boldsymbol{v}} - \mathring{D}_{\boldsymbol{v}}\mathring{D}_{\boldsymbol{u}} - \mathring{D}_{[\boldsymbol{u},\boldsymbol{v}]}, \\ \boldsymbol{\rho}(\boldsymbol{u}, \boldsymbol{v}) &= D_{\boldsymbol{u}}D_{\boldsymbol{v}} - D_{\boldsymbol{v}}D_{\boldsymbol{u}} - D_{[\boldsymbol{u},\boldsymbol{v}]}. \end{aligned} \tag{28}$$

Also, the Riemann curvature tensors of those connections are respectively the objects:

$$\mathring{\boldsymbol{R}}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v},\boldsymbol{\alpha}) := \boldsymbol{\alpha}(\mathring{\boldsymbol{\rho}}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}),$$
$$\boldsymbol{R}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v},\boldsymbol{\alpha}) := \boldsymbol{\alpha}(\boldsymbol{\rho}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}). \tag{29}$$

Moreover, the components of the curvature tensors relative in the appropriated coordinate basis associated to the coordinates $\langle x^\mu \rangle$ covering U are:

$$\mathring{\boldsymbol{R}}(\boldsymbol{e}_\mu,\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta,\boldsymbol{\vartheta}^\lambda) := \mathring{R}_{\mu\alpha\beta\cdot}^{\cdot\,\cdot\,\cdot\,\lambda}$$
$$\boldsymbol{R}(\boldsymbol{e}_\mu,\boldsymbol{e}_\alpha,\boldsymbol{e}_\beta,\boldsymbol{\vartheta}^\lambda) := R_{\mu\alpha\beta\cdot}^{\cdot\,\cdot\,\cdot\,\lambda}. \tag{30}$$

We get after some trivial (but tedious algebra as in the last section) [10] that

$$R_{\mu\alpha\beta\cdot}^{\cdot\,\cdot\,\cdot\,\lambda} = \mathring{R}_{\mu\alpha\beta\cdot}^{\cdot\,\cdot\,\cdot\,\lambda} + J_{\mu[\alpha\beta]\cdot}^{\cdot\,\cdot\,\cdot\,\lambda} \tag{31}$$

where

$$J_{\mu\alpha\beta\cdot}^{\cdot\cdot\cdot\lambda} = D_\alpha K_{\beta\mu\cdot}^{\cdot\cdot\lambda} - K_{\beta\sigma\cdot}^{\cdot\cdot\lambda} K_{\alpha\mu\cdot}^{\cdot\cdot\sigma}$$
$$J_{\mu[\alpha\beta]\cdot}^{\cdot\,\cdot\,\cdot\,\lambda} = J_{\mu\alpha\beta\cdot}^{\cdot\,\cdot\,\cdot\,\lambda} - J_{\mu\beta\alpha\cdot}^{\cdot\,\cdot\,\cdot\,\lambda}. \tag{32}$$

6 Geometry of a Manifold where $\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} = -\Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}$ in some Coordinates $\langle x^\mu \rangle$ Covering $U \subset M$

Does the condition $\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} = -\Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}$ implies that $\mathring{\Gamma}_{\mu\lambda\cdot}^{\cdot\cdot\rho} = 0$? Of course, *not in general*. Let us see the reason for that. From Eq.(5) we see that all that is necessary for the validity of $\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} = -\Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}$ is that in the coordinate system where $\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} = -\Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}$ we have

$$\mathring{\Gamma}_{\mu\lambda\cdot}^{\cdot\cdot\rho} = -\frac{1}{2} S_{\mu\lambda\cdot}^{\cdot\cdot\rho} \tag{33}$$

Observe moreover that when $\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} = -\Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}$ and besides that we have also $S_{\mu\lambda\cdot}^{\cdot\cdot\rho} = 0$, then it follows from Eqs.(8) that $\partial_\mu g_{\nu\lambda} = 0$, i.e., the $g_{\nu\lambda}$ are *constant functions* of the coordinates. Then, taking into account the Lorentz signature of the metric we can introduce coordinates in some open set intersecting U such that the matrix with entries $g'_{\nu\lambda}$ is the diagonal matrix diag$(1,-1,-1,-1)$.

Thus we see that taking into account Eq.(23) when $\Gamma_{\mu\lambda\cdot}^{\cdot\cdot\rho} = -\Gamma_{\lambda\mu\cdot}^{\cdot\cdot\rho}$ we have that $\mathring{\Gamma}_{\mu\lambda\cdot}^{\cdot\cdot\rho} = 0$ only if $S_{\mu\lambda\cdot}^{\cdot\cdot\rho} = 0$.

However, it is a good idea to keep in mind that even in that case the Riemann curvature tensor of the connection Riemann-Cartan connection D is not null in general. Indeed, from Eqs.(31) and (32) it follows that

$$\begin{aligned} R_{\mu\alpha\beta\cdot}^{\cdot\,\cdot\,\cdot\,\lambda} &= J_{\mu[\alpha\beta]\cdot}^{\cdot\,\cdot\,\cdot\,\lambda} \\ &= D_\alpha K_{\beta\mu\cdot}^{\cdot\cdot\lambda} - K_{\beta\sigma\cdot}^{\cdot\cdot\lambda} K_{\alpha\mu\cdot}^{\cdot\cdot\sigma} - D_\beta K_{\alpha\mu\cdot}^{\cdot\cdot\lambda} + K_{\alpha\sigma\cdot}^{\cdot\cdot\lambda} K_{\beta\mu\cdot}^{\cdot\cdot\sigma} \end{aligned} \tag{34}$$

Finally, we ask: what is a sufficient condition for $R_{\mu\alpha\beta\cdot}^{\cdot\cdot\cdot\lambda} = 0$?

The condition is the existence of four *parallel vector fields* defined on all M such that they are basis for each $T_x M$ (the tangent space at $x \in M$). A set of parallel vector fields $\{\mathbf{X}_a\}$, $a = 0, 1, 2, 3$ is by definition one such that $D_{\mathbf{X}_a}\mathbf{X}_b = 0$ for all $a, b = 0, 1, 2, 3$. A space with this property is called parallelizable and in the case where it has Lorentzian metric is known as *Weintzbock spacetime*.

7 Final Remarks

In his note 122 [4] MWE states as 'theorem' that *any* connection must be anti-symmetric. From Eq.(6) above it is obvious that this statement is simply ***wrong***. That error simply invalidates almost all of his statements presented in his series of papers on 'ECE theory'. And indeed, it is well known that some (if not all) of those papers are full of very serious errors, including one that MWE calls the '*dual Bianchi identity*', a non sequitur that leads him to claim that Einstein's equations are mathematically wrong! Of course they are not. For more details on this particular issue see [11]. For a discussion of some another MWE serious flaws see also [9, 7, 8, 2].

Those sad facts are being presented here because despite the criticisms quoted above that simply show that ECE theory is a nonsequitur, MWE recently found support from a british publisher[4] to launch a journal: *Journal of Foundations of Physics and Chemistry* which will publish all the papers he and others authored on *ECE*. Before you order such a journal, please give a read with attention on this pedagogical and *free* note and also the *free* references quoted below.

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